Note Title 11/17/2006

Exam Fri. Dec 1

No HW next week.

Graded HW on my Door

Wednesday 11/22 review session

11/17/06

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Sep. of variables for $\nabla^2\psi=0$ gave the following ODE for $P(\theta)$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right)+\left[\ell(\ell+1)-\frac{m^2}{\sin^2\theta}\right]P=0$$

or, using $x=\cos\theta$ $\left[\frac{1}{\sin\theta}\frac{d}{d\theta}=-\frac{d}{dx}\right]$

$$\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right]+\left[\ell(\ell+1)-\frac{m^2}{1-x^2}\right]P=0$$

Equivalent forms of Legendre's equation.

Solutions $\equiv P_{\ell m}(x)$

associated Legendre functions

important special case $m=0$, Then the φ part of the solution is

$$Q(\varphi) = 1$$

no φ dependence

then we have

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right) + \ell(\ell+1)P(x) = 0$$

Now P has only one index

$$P_\ell(x) = P_\ell(\cos\theta)$$

ordinary legendre funct.

Turns out

$$P_\ell(x) = \frac{1}{2^\ell \ell!} \frac{d^\ell}{dx^\ell} (x^2-1)^\ell$$

$$P_{\ell m}(x) = (-1)^m (1-x^2)^{m/2} \frac{d^m}{dx^m} P_\ell(x)$$

$$P_0 = 1$$

$$P_1(x) = x \qquad P_1(\cos\theta) = \cos\theta$$

$$P_2 = \frac{1}{2}(3x^2-1)$$

$$\vdots$$

$$x \in [-1, 1]$$

recall our definition of the inner product of 2 function defined on $[-\ell, \ell]$

$P_\ell(x)$ are an orthogonal set of functions on $[-1,1]$

Completeness any function on $[-1,1]$ can be written as a superposition of $P_\ell(x)$:

$f(x)$ on $[-1,1]$

$$f(x) = \sum_{\ell=0}^{\infty} A_\ell P_\ell(x)$$

$$(P_m, f) = \sum_{\ell=0}^{\infty} A_\ell \underbrace{(P_m, P_\ell)}_{\frac{2}{2\ell+1}\delta_{m\ell}}$$

$$\boxed{(P_m, f) = \frac{2}{2m+1} A_m}$$

$$(f, g) = \frac{1}{2\ell} \int_{-\ell}^{\ell} f(x) g(x)\, dx$$

So $$(P_0, P_1) = \frac{1}{2} \int_{-1}^{1} 1 \cdot x \, dx$$

$$= \frac{1}{2} \left[x^2/2 \Big|_{-1}^{1} \right] = 0$$

$$(P_0, P_2) = \frac{1}{2} \int_{-1}^{1} 1 \cdot \frac{1}{2}(3x^2 - 1)\, dx$$

$$= \frac{1}{4} \int_{-1}^{1} 3x^2 \, dx - \int_{-1}^{1} dx$$

$$\frac{1}{4}\left(3 \frac{x^3}{3} \Big|_{-1}^{1}\right) - \frac{1}{4}\left(x \Big|_{-1}^{1}\right)$$

$$= \frac{1}{2} - \frac{1}{2} = 0$$

The Legendre functions are orthog.

in fact, one can show that

$$\int_{-1}^{1} P_\ell(x) P_m(x)\, dx = \frac{2}{2\ell+1}\, \delta_{\ell m}$$

$$\boxed{(P_\ell, P_m) = \frac{\delta_{\ell m}}{2\ell+1}}$$

important digression

Power Series Solution to diff Eq.

$$\boxed{y'' + y = 0}$$

Pretend you don't know how to solve this

Let $\quad y(x) = \sum_{n=0}^{\infty} a_n x^n$

i.e. $\quad y(x) = a_0 + a_1 x + a_2 x^2 + \cdots$

$$y'(x) = a_1 + 2a_2 x + \cdots$$

$$= \sum_{?} a_n n x^{n-1}$$

$$= \sum_{n=0}^{\infty} a_{n+1} (n+1) x^n$$

$$y'' = 2a_2 + 3\cdot 2\cdot a_3 x + \cdots$$

$$= \sum_{?} a_{n+1}(n+1)\,n\,x^{n-1}$$

$$= \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)\,x^{n}$$

So $y'' + y = 0 \Rightarrow$

$$\sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)\,x^{n} + a_n x^n = 0$$

$$\Rightarrow \underbrace{\sum_{n=0}^{\infty} \left[a_{n+2}(n+2)(n+1) + a_n\right] x^{n} = 0}$$

can only be true for arbitrary x if

$$a_{n+2}(n+2)(n+1) + a_n = 0$$

$$\Rightarrow \quad a_{n+2} = \frac{-a_n}{(n+2)(n+1)}$$

recursion relation

Suppose we set $a_0 = 1$

$$a_2 = -\frac{a_0}{2} = -\tfrac{1}{2}$$

but a_1 is not determined

So take $a_1 = 0$ for simplicity .... continue

$$a_3 = \frac{-a_1}{3 \cdot 2} = 0$$

$$a_4 = \frac{-a_2}{4 \cdot 3} = \frac{1}{4 \cdot 3 \cdot 2}$$

$$\vdots$$

$$a_{2N} = \frac{(-1)^N}{(2N)!}$$

So $$y(x) = \sum_{N=0}^{\infty} \frac{(-1)^N}{(2N)!} x^{2N}$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \cdots$$

$$= \cos(x)$$

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on the other hand if we choose

$a_0 = 0$, $a_1 = 1$ we

get a different solution

$$a_{n+2} = \frac{-a_n}{(n+2)(n+1)}$$

$$a_2 = \frac{-a_0}{4 \cdot 3} = 0$$

$$a_3 = \frac{-a_1}{3 \cdot 2} = -\frac{1}{3 \cdot 2}$$

$$a_4 = 0$$

$$a_5 = \frac{-a_3}{5 \cdot 4} = \frac{1}{5!}$$

$$\vdots$$

$$y(x) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}$$

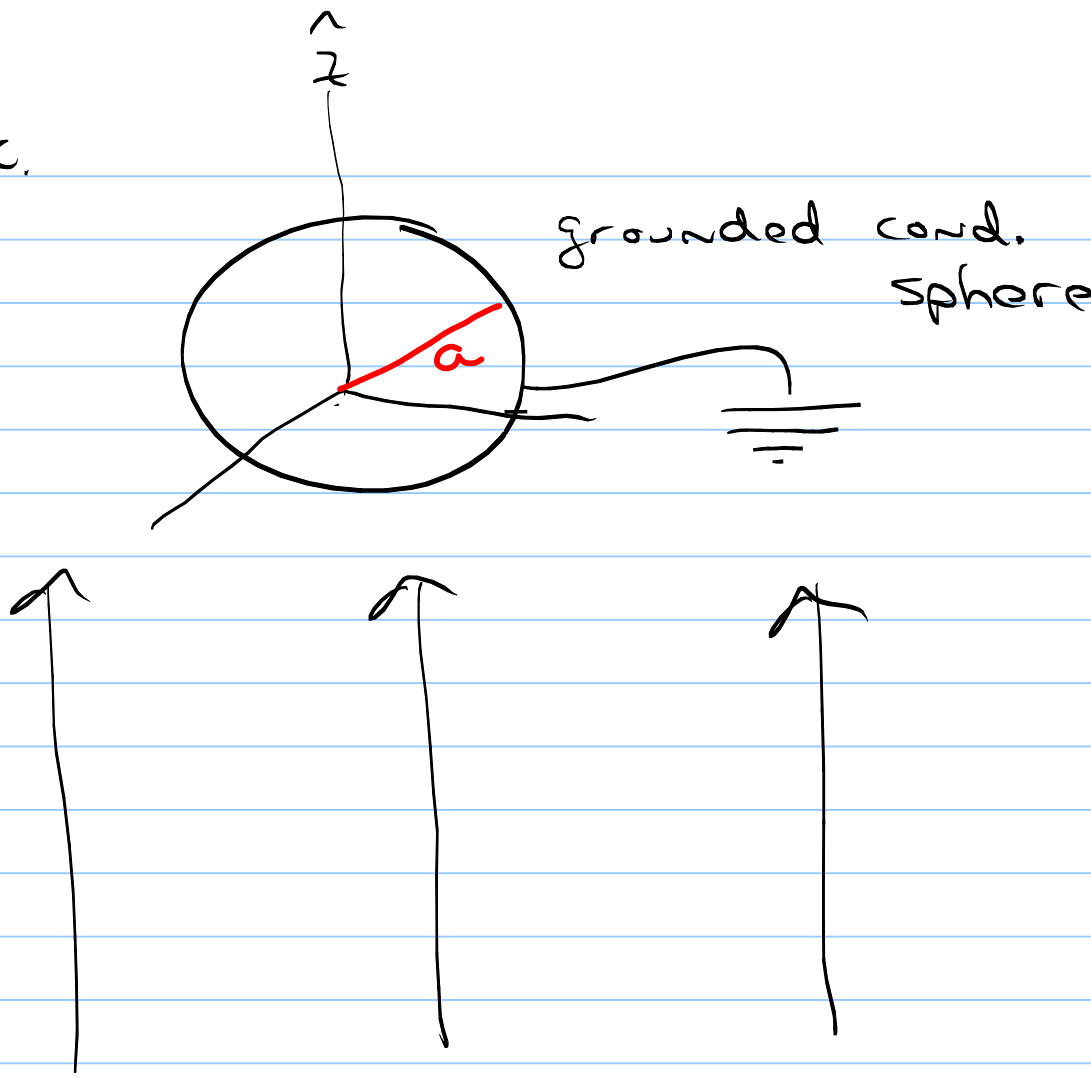
$= \sin(x)$

two independent sol'ns

so $\boxed{y(x) = c_1 \cos x + c_2 \sin x}$

we knew this, but now we see how we can get solutions via power series.

Ex.



$$\vec{E} = E_0 \hat{z}$$

Axial symmetry $\Rightarrow m=0$

So our solution to $\nabla^2 V = 0$

$Y_{\ell m}(\theta,\phi)$

$$V(r,\theta,\phi) = \sum \left(A_{\ell m} r^{\ell} + B_{\ell m} r^{-(\ell+1)}\right) Y_{\ell m}$$

$$\Downarrow$$

$$V(r,\theta) = \sum_{\ell=0}^{\infty} \left(A_\ell r^{\ell} + B_\ell r^{-(\ell+1)}\right) P_\ell$$

$P_\ell(\cos\theta)$

Since sphere is conducting

BC ① $V(r=a,\theta) = 0$

$$= \sum_{\ell=0}^{\infty} \left(A_\ell a^{\ell} + B_\ell a^{-(\ell+1)}\right) P_\ell = 0$$

BC ② Since $\bar{E}$ as $r \to \infty = E_0 \hat{z}$

$$\Rightarrow V(r\to\infty,\theta) = -E_0 z$$