Optics FAQ

Relations to know:

$k = k_0 n$

$v_{ph} = c/n$

$\omega = k_0 c = k\, v_{ph}$. Frequency is always constant through the medium. For n > 1, $v_{ph} < c$, so to keep constant frequency, λ *in medium* is smaller (larger *k*)

$v_g = \frac{d\omega}{dk}$, but in optics $k = k_0 n(\omega) = \omega\, n(\omega)/c$, so we calculate group velocity by calculating $\frac{1}{v_g(\omega)} = \frac{dk(\omega)}{d\omega}$.

Refraction

- $n \sin\theta$ is constant through the medium (Snell's law). If *x* is the direction along the surface and in the plane of incidence, then this means $k_x = k_0 n \sin\theta$ is also continuous. Physically, this results from the fact that all the wavefronts are continuous (constant phase) along the boundaries. This is true both in layered materials, waveguides, evanescent waves, and where the index varies continuously.
- There is *no refraction at normal incidence*.
- When tracing rays through an interface, draw the surface normal. When the ray is going from low index to high, refraction is *toward* the normal (never past the normal). When the ray is going from high index to low, refraction is *away* from the normal.

TIR:

- At the critical angle for TIR, the refracted ray is along the surface ($\theta = 90^o$), so $n_1 \sin\theta_c = n_2$ for a ray going from medium 1 to 2.
- For total internal reflection (TIR), the ray must be going from a high refractive index to low ($n_1 > n_2$).
- In the common situation where $n_2 = 1$ (air or vacuum), the critical angle (from the normal) is *smaller* if the incident medium has a *bigger* index (n_1). In other words, there is a wider range of angles for TIR for larger index.

Absorption:

In the present context, absorption means taking energy out of the beam. In practice, this could mean it is left in the medium (as in heating) or scattered out of the medium.

- If the refractive index is purely imaginary $n = i n_I$, the wave is exponentially damped without oscillation. In this case, there is no power absorbed, all is reflected. Examples: evanescent wave in TIR, reflection from a collisionless plasma with input frequency less than the plasma frequency, reflection of a wave below the cutoff frequency of a metal waveguide.
- With our sign convention, $E \propto \exp[i(kz - \omega t)]$, a positive n_I is a damped wave, negative is growing (as when there is gain). Always check to make sure your

signs are right: some texts other than ours (such as Hecht) use the other convention.
- If the medium has a complex index, with non-zero n_R and n_I, the wave is partly absorbed in the medium.

Fresnel equations
P-polarization: *E*-field is in the plane of incidence (plunge).
S-polarization: *E*-field is normal to the plane of incidence (skim). *E* is along the surface here.
If the incident light is linearly polarized as *S* or *P*, the polarization *does not change* upon reflection or refraction.

Magnitude of reflections:
P-polarized light generally reflects less than *S*-polarized light. Remember that it is *P*-polarized light that has the Brewster angle.
Higher index difference at an interface generally leads to more reflected light.

Brewster angle: $\tan\theta_B = \frac{n_2}{n_1}$

- At this angle, there is complete transmission of *P*-polarized light. *S*-polarized light *transmits* and *reflects* at the Brewster angle.
- So with unpolarized light incident, the reflected light is completely S-polarized, but the transmitted light is mixed.
- The *internal* and *external* Brewster angles are different, but light incident at the Brewster angle on a window with parallel surfaces will automatically be at the Brewster angle at *both* surfaces.

Phase shifts:
- If the incident light has a polarization that isn't already *S* or *P* (relative to the local surface), then write the E-field as a sum of *S* and *P* vector components. The resulting polarization state may in general change due to relative phase shifts.
- *Except for TIR*, all phase shifts at interfaces are either 0 or π (just a sign change).
- Near normal incidence, there is a sign change when the index transition is from low to high, and no sign change from high to low. This is the convention in Hecht, and originates from the initial choice of orientation of the fields. Whichever convention is used, there is a sign change in one case and not the other.

Composing functions:
There are certain functions that we use all the time, and you must be able to write down the functions in such a way that you can shape and position them according to the problem.
You should also be able to tell how the patterns scale with their arguments.
Here are some common ones. Let *s* be a dimensionless variable.
- $\cos(s), \sin(s)$ These are the same: just a π/2 phase shift off: $\cos(s) = \sin(s + \pi/2)$
- Exponential: e^{is} , e^{-s}

- Gaussian: e^{-s^2}
- $\mathrm{sinc}(s) = \sin(s)/s$

You should know how to shift and scale these, for example $Ae^{-(x-x_0)^2/\Delta x^2}$ is a Gaussian profile with peak amplitude A, centered on $x = x_0$, with a $1/e$ half width of Δx.

Complex variables:

You should know the complex representations of the sine and cosine functions:

$\cos z = \frac{1}{2}\left(e^{iz} + e^{-iz}\right)$, $\sin z = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right)$ $\qquad e^{iz} = \cos z + i \sin z$

And the identities: $z + z^* = 2\,\mathrm{Re}(z)$ $\quad z - z^* = 2\,\mathrm{Im}(z)$

You should also know the Taylor expansions of these functions out to second order in z.

Fourier transforms:

- You should be able to prove the majority of the identities and transform pairs, or at least understand where they come from.
- The main reason I use the transform sheet is to keep track of signs and factors of 2π.
- It helps a great deal to have a visual association of each of the functions with its transform pair.
- Once you know how to actually integrate to do the Fourier transforms, learn how to use the theorems, transform identities to simplify the transform. Often the whole problem can be done without any integration, just application of the transform ID's and pairs. In real life problems, you can often reduce the transform to one part that must be integrated.
- Except for a few simple cases involving delta functions, analytic (as opposed to numeric) convolutions are harder to do than Fourier transforms. You can use the convolution theorem to avoid difficult convolutions. Even if you may not actually perform the convolution, it is very important to understand what the convolution operation does to understand the problem physically.
- In our course, the sign convention for the forward and inverse transforms is reversed when we go from t, w space to x, β_x space. This sign reversal has its origins in the opposite sign of $k\,z$ and ωt in a forward-propagating plane wave $E \propto \exp\left[i\left(k\,z - \omega t\right)\right]$

Diffraction

- Conceptually, diffraction can be described as resulting from the superposition of a dense array of point sources across the aperture.
- Mathematically, this superposition is represented by an integral which is the convolution of the spherical waves (from the point sources) with the field that makes it through the aperture.
- In the near field, this integral is the Fresnel integral (hard to do analytically).
- In the far field, the integral is the Fraunhofer integral, which is connected to the Fourier transform using the concept of spatial frequency described above.
- The diffraction integral can be thought of as a calculation that just gives the field at a plane farther downstream. Therefore, the construction of integrating over $E_{in}(x, y)A(x, y)$, that is, the product of an input field and an aperture function, is artificial (albeit helpful). The integral is over whatever the field is at the starting plane: there does not have to be an actual aperture present.

- The simplest scaling to remember about diffraction from an aperture (distance d across) is the diffraction angle: $\theta_d \approx \lambda / d$. This expression is true in the far field for any confined beam. Think of this as an uncertainty relation: confining the beam spatially leads to a spread in its direction.
- Other far-field diffraction patterns follow the Fourier transform relations and scale in the same way. For example, finer features at the input plane diffract to larger angles in the far field.

Dispersing devices (diffraction gratings, Fabry-Perot interferometers)
You should be familiar with the layout of a basic spectrometer: entrance slit, collimating mirror, diffraction grating, focusing mirror, exit slit.
Dispersion: the wavelength or frequency that is transmitted through a device will in general be dependent on several parameters. The dispersion is calculated by finding the derivative of the transmitted wavelength with respect to that variable. For example, the wavelength dispersion of a diffraction grating with respect to diffraction angle is $\partial\lambda / \partial\theta_m$.
Where in this case the grating equation is used for $\lambda(\theta_m)$.
Resolution: any instrument that can separate signals of different wavelengths will have a finite resolution (called the instrument width). Expressed in terms of wavelength, the resolution $\Delta\lambda$ corresponds to the minimum resolvable wavelength difference. With a single frequency input, the transmitted signal as a function of one of the variables will have some spread to it. You should understand the fundamental limits to resolution for the spectrometer and the Fabry-Perot:
- Spectrometer: the resolution is limited by the size of the grating. Single frequency light is reflected from the surface and is brought to a focus by the mirror. The size of the diffraction spot, represented in terms of the diffracted angle, can be related to the wavelength resolution.
- Fabry-Perot: The *finesse* determines the width of the resonance line. Calculating the FWHM of the transmitted line from the Fabry-Perot transmission function is straightforward.

Useful relations to connect variations in different quantities: $\frac{\Delta\lambda}{\lambda} = \frac{\Delta\nu}{\nu} = \frac{\Delta\omega}{\omega}$.
Free spectral range: if for a given setting of your dispersing device, a frequency ν_1 passes in the m^{th} order, there will be another frequency ν_2 that will pass in the $(m+1)^{th}$ order. The free spectral range is then $\Delta\nu=|\nu_2-\nu_1|$. If the input is restricted in bandwidth to less than the FSR, there will not be any confusion about what frequencies are being measured. The FSR is quite large for gratings, and quite small for a Fabry-Perot.

Interference
- Know how to calculate the intensity of the combination of two waves, including phase and polarization.
- Several sources of phase differences: optical path (variations in distance and/or index), reflection phase shifts.
- *Coherence time*: $\tau_c = 1/\Delta\nu$, i.e., the inverse of the bandwidth of the source. The temporal coherence length is just $c\,\tau_c$, the distance that corresponds to the coherence

time. A finite coherence time results from the washing out of interference as the different frequency components slip in phase at delays away from zero.

- *Spatial coherence:* represents a measure of the phase relationship between different portions of a wavefront.

Waveguides

- Geometric/ray picture: rays are confined by total internal reflection. There are well-defined angles at which there is constructive interference between waves reflecting from opposite surfaces.
- EM calculation: dielectric profiles are like the inverse of energy potential profiles ($U(x)$) encountered in quantum mechanics. Specifically, $-n^2(x)k_0^2 \approx U(x)$. In the waveguide, the eigenvalue is the forward wavenumber, k_z^2.
- For a waveguide axis in the z-direction, the wave takes the separable form $E(x,y,z,t) = E_0(x,y)e^{i(k_z z - \omega t)}$.
- As in refraction and TIR, the vector relationship between the components of the k-vector, $n^2(x)k_0^2 = k_x^2(x) + k_z^2$, holds true, even in regions where k_x is imaginary. This relation provides the connection between the transverse wavenumber k_x and the propagation wavenumber k_z.
- The guided wave has planar (flat) wavefronts, even though the wave is modulated in amplitude in the transverse direction.
- The velocity of the wave is connected to the propagation wavenumber k_z: $v_{ph} = \omega / k_z$, and $v_g = d\omega / dk_z$.
- The *higher-order modes* propagate with a *higher phase velocity*: they correspond to rays that have a steeper angle with respect to the waveguide axis. The group velocity generally behaves in the opposite manner: higher modes have a slower v_g.
- When more than one spatial mode is propagating, there is interference between the modes, and the interference pattern varies along z owing to the different modal phase velocities.
- When light leaves a waveguide, it propagates in free space according to the rules of diffraction.