MACS 332A - Fall 2008

NAME:__________________________________ **Exam I**

In order to receive full credit, SHOW ALL YOUR WORK. Full credit will be given only if all reasoning and work is provided. Please enclose your final answers in boxes.

1. Given the system of equations,

$$\begin{array}{rcrcrcr} & & x_2 & + & 3x_3 & = & 8, \\ 2x_1 & & & - & 4x_3 & = & -10, \\ -2x_1 & - & 2x_2 & - & 2x_3 & = & -6 \\ 3x_1 & + & 2x_2 & & & = & 1 \end{array}$$

(a) (12 points) Solve the system by using row operations to get an augmented matrix in reduced row-echelon form. Write the solution in parametric-vector form.

$$\begin{bmatrix} 0 & 1 & 3 & 8 \\ 2 & 0 & -4 & -10 \\ -2 & -2 & -2 & -6 \\ 3 & 2 & 0 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 2 & 0 & -4 & -10 \\ 0 & 1 & 3 & 8 \\ 0 & -2 & -6 & -16 \\ 0 & 2 & 6 & 16 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & -2 & -5 \\ 0 & 1 & 3 & 8 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \mathbf{x} = x_3 \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} + \begin{bmatrix} -5 \\ 8 \\ 0 \end{bmatrix}$$

(b) (5 points) Find the solution for the homogeneous system corresponding to the given system. Describe the geometric relationship between the homogeneous solution and the nonhomogeneous solution found in part (a)

$$\mathbf{x} = x_3 \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$$

The non-homogeneous solution can be represented by a shift (specifically by $\begin{bmatrix} -5 \\ 8 \\ 0 \end{bmatrix}$) from the homogeneous solution.

(c) (5 points) Do the columns of the coefficient matrix for the system span $\mathbb{R}^4$? Justify your answer.

The columns of the given coefficient matrix do not span $\mathbb{R}^4$ because there is not a pivot in every row.

2. (7 points) Suppose that the coefficient matrix corresponding to a linear system is a 5×7 matrix and has 4 pivot columns. How many pivot columns does the augmented matrix have if the linear system is inconsistent?
   The augmented matrix will have 5 pivot columns since, if the system is inconsistent, the last column in the augmented matrix will also be a pivot column.

3. Construct a set of four vectors in $\mathbb{R}^4$ that are
   (a) (6 points) linearly independent
   As an example,

$$\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix}$$

   (b) (6 points) linearly dependent Any set which
      i. Include the $\mathbf{0}$ vector
      ii. Include vectors that are multiples of each other or a vector that is a linear combination of thte others

4. (12 points) Let $A = \begin{bmatrix}-3 & 1 & 4\\6 & 0 & -2\\-6 & 2 & 8\end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix}3\\-8\\6\end{bmatrix}$. Solve the equation $A\mathbf{x} = \mathbf{b}$ using an LU factorization.

$$\begin{bmatrix}-3 & 1 & 4\\6 & 0 & -2\\-6 & 2 & 8\end{bmatrix} \rightsquigarrow \begin{bmatrix}-3 & 1 & 4\\0 & 2 & 6\\0 & 0 & 0\end{bmatrix}$$

Thus, the LU-factorization of A is

$$A = \begin{bmatrix}1 & 0 & 0\\-2 & 1 & 0\\2 & 0 & 1\end{bmatrix} \begin{bmatrix}-3 & 1 & 4\\0 & 2 & 6\\0 & 0 & 0\end{bmatrix}$$

Using this factorization to solve the system gives

$$\begin{bmatrix}1 & 0 & 0\\-2 & 1 & 0\\2 & 0 & 1\end{bmatrix} \mathbf{y} = \begin{bmatrix}3\\-8\\6\end{bmatrix} \Rightarrow \mathbf{y} = \begin{bmatrix}3\\-2\\0\end{bmatrix}$$

$$\begin{bmatrix}-3 & 1 & 4\\0 & 2 & 6\\0 & 0 & 0\end{bmatrix} \mathbf{x} = \begin{bmatrix}3\\-2\\0\end{bmatrix} \Rightarrow \mathbf{x} = x_3 \begin{bmatrix}1/3\\-3\\1\end{bmatrix} + \begin{bmatrix}-4/3\\-1\\0\end{bmatrix}$$

5. (10 points) Let A and B be invertible $n \times n$ matrices. Show that $(A^{-1}(B^{-1})^T)^T$ is the inverse of A^TB.
For $(A^{-1}(B^{-1})^T)^T$ to be the inverse of A^TB, it must be the case that

$$\left[(A^{-1}(B^{-1})^T)^T\right]\left[A^TB\right] = I$$

So,

$$\begin{aligned}\left[(A^{-1}(B^{-1})^T)^T\right]\left[A^TB\right] &= \left[((B^{-1})^T)^T(A^{-1})^T\right]\left[A^TB\right]\\ &= \left[((B^{-1})^T)^T\right]((A^{-1})^TA^T)B\\ &= \left[((B^{-1})^T)^T\right](AA^{-1})^T)B\\ &= \left[((B^{-1})^T)^T\right]B\\ &= B^{-1}B\\ &= I\end{aligned}$$

Similarly for

$$\left[A^TB\right]\left[(A^{-1}(B^{-1})^T)^T\right] = I$$

6. (12 points) Let $A = \begin{bmatrix} 0 & 5 & 0 & 2 & 0 \\ 0 & 2 & 0 & -3 & -1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -3 & 0 & 0 \\ 5 & 5 & 0 & 0 & 0 \end{bmatrix}$. Find $\det A$.

$$\begin{aligned}\det A &= (-1)(-3)\begin{vmatrix} 0 & 5 & 2 & 0 \\ 0 & 2 & -3 & -1 \\ 1 & 0 & 1 & 0 \\ 5 & 5 & 0 & 0 \end{vmatrix}\\ &= (-1)(-3)(-1)\begin{vmatrix} 0 & 5 & 2 \\ 1 & 0 & 1 \\ 5 & 5 & 0 \end{vmatrix}\\ &= (-1)(-3)(-1)\left[(-1)(1)\begin{vmatrix} 5 & 2 \\ 5 & 0 \end{vmatrix} + (-1)(1)\begin{vmatrix} 0 & 5 \\ 5 & 5 \end{vmatrix}\right]\\ &= (-1)(-3)(-1)\left[-10 + -25\right]\\ &= -105\end{aligned}$$

7. (11 points) Let A be a 4×5 matrix, let $\mathbf{y}_1$ and $\mathbf{y}_2$ be vectors in $\mathbb{R}^4$, and let $\mathbf{w} = \mathbf{y}_1 + \mathbf{y}_2$. Suppose $\mathbf{y}_1 = A\mathbf{x}_1$ and $\mathbf{y}_2 = A\mathbf{x}_2$ for some vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ in $\mathbb{R}^5$. Explain why the system $A\mathbf{x} = \mathbf{w}$ is consistent.
Since $\mathbf{y}_1 = A\mathbf{x}_1$ and $\mathbf{y}_2 = A\mathbf{x}_2$, we know that

$$\mathbf{w} = \mathbf{y}_1 + \mathbf{y}_2 = A\mathbf{x_1} + A\mathbf{x_2} = A(\mathbf{x}_1 + \mathbf{x}_2)$$

Therefore, the system $A\mathbf{x} = \mathbf{w}$ has, as a solution, $\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2$ and therefore is consistent.

8. (12 points) Given the system of equations

$$\begin{aligned} x_1 + 3x_2 &= k \\ -2x_1 + hx_2 &= 8 \end{aligned}$$

(a) find all values of h and k fo which the system has a unique solution.

$$\begin{bmatrix} 1 & 3 & k \\ -2 & h & 8 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 3 & k \\ 0 & h+6 & 2k+8 \end{bmatrix}$$

For the system to have a unique solution, the first two columns must be pivot columns. Therefore, $h \neq 6$. In this case, k can be any real number.

$$h \neq 6,\ k \in \mathbb{R}$$

(b) find all values of h and k for which the system has infinitely many solutions.
Given the row reduction above, for the system to have infinitely many solutions, the system must have a free variable. Therefore, $h = -6$. However, the system must also be consistent, and thus not have a pivot in the last column. Therefore, $k = -4$.

$$h = -6, \text{ and } k = -4$$