

You must show all of your work to get full credit.

$$\frac{dx}{dt} = x^2 - 2xy + 2x \approx x(x - 2y + 2) = f(x, y)$$

1. Consider the system:

EP:

$$(0,0), (0,-8), \\ (-2,0), (-6,-2)$$

a. Find all equilibrium points $\frac{dx}{dt} = 0 + \frac{dy}{dt} = 0$

$$x(x - 2y + 2) = 0 \quad y(y + x + 8) = 0 \\ \underline{x=0} \quad \underline{x=2y-2} \quad \underline{x=0}: y(y+8)=0 \quad \underline{x=2y-2}: y(y+2)-2+8=0 \\ \underline{y=0}, \underline{y=-8} \quad \underline{y=0} \quad \underline{y=-6} \\ \underline{y=-2}$$

b. Using the Jacobian matrix, classify all equilibrium points.

$$J(x, y) = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} = \begin{pmatrix} 2x - 2y + 2 & -2x \\ y & 2y + x + 8 \end{pmatrix}$$

$$(0,0): J(0,0) = \begin{pmatrix} 2 & 0 \\ 0 & 8 \end{pmatrix} = A$$

$$\det \begin{pmatrix} 2-\lambda & 0 \\ 0 & 8-\lambda \end{pmatrix} = 0$$

$$(2-\lambda)(8-\lambda) = 0$$

$$\lambda_1 = 2, \lambda_2 = 8$$

| source

$$(0,-8): J(0,-8) = \begin{pmatrix} 18 & 0 \\ -8 & -8 \end{pmatrix} = A$$

$$\det \begin{pmatrix} 18-\lambda & 0 \\ -8 & -8-\lambda \end{pmatrix} = 0$$

$$(18-\lambda)(-8-\lambda) = 0$$

$$\lambda_1 = 18, \lambda_2 = -8$$

| Saddle

$$(-2,0): J(-2,0) = \begin{pmatrix} -2 & 4 \\ 0 & 6 \end{pmatrix} = A$$

$$\det \begin{pmatrix} -2-\lambda & 4 \\ 0 & 6-\lambda \end{pmatrix} = 0$$

$$(-2-\lambda)(6-\lambda) = 0$$

$$\lambda_1 = -2, \lambda_2 = 6$$

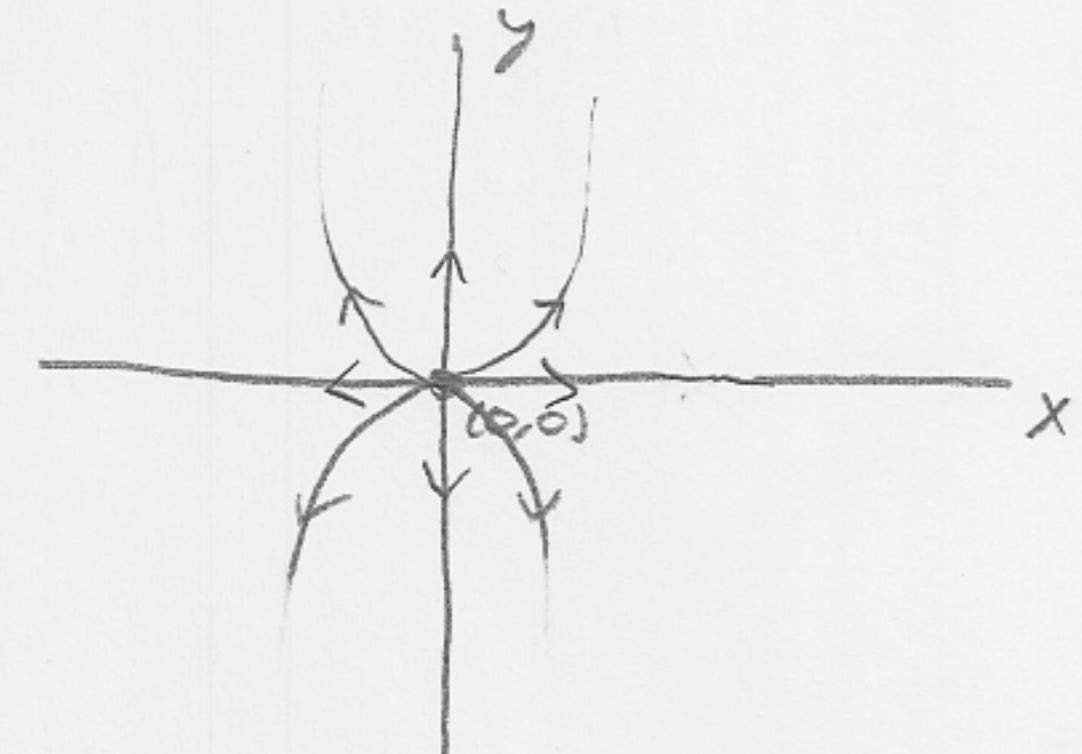
| Saddle

c. For each equilibrium point sketch the phase portrait of the system near that point. Sketch each of these graphs separately.

$$(0,0): \\ \lambda_1 = 2: (A - \lambda_1 I) \vec{v}_1 = \vec{0} \\ \begin{pmatrix} 0 & 0 \\ 0 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ 6y_1 = 0, \underline{y_1 = 0}, x_1 = \infty \\ \vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\lambda_2 = 8: (A - \lambda_2 I) \vec{v}_2 = \vec{0} \\ \begin{pmatrix} -6 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ -6x_2 = 0, \underline{x_2 = 0}, y_2 = \infty \\ \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\vec{Y}(t) = k_1 e^{2t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + k_2 e^{8t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$



Soln Linear : Approx. Nonlinear

(See Attached for Rest.)

p.2

1. (c) (cont.)

(0, -8):

$$\lambda_1 = 18 : (A - \lambda_1 I) \vec{v}_1 = \vec{0}$$

$$\begin{pmatrix} 0 & 0 \\ -8 & -26 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$-8x_1 - 26y_1 = 0$$

$$y_1 = \frac{-8x_1}{26} = \frac{-4x_1}{13}$$

$$x_1 = \alpha, y_1 = \frac{-4\alpha}{13}$$

$$\vec{v}_1 = \begin{pmatrix} 13 \\ -4 \end{pmatrix}$$

$$\lambda_2 = -8 : (A - \lambda_2 I) \vec{v}_2 = \vec{0}$$

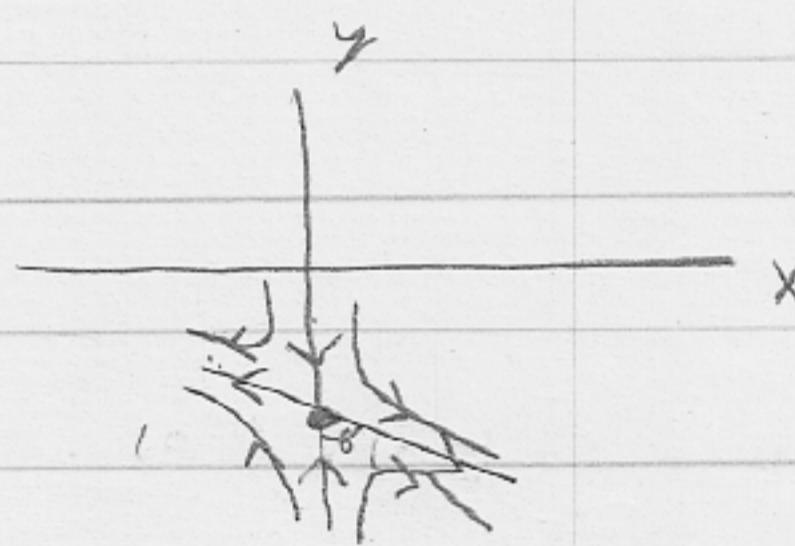
$$\begin{pmatrix} 26 & 0 \\ -8 & 0 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$26x_2 = 0,$$

$$x_2 = 0, y_2 = \alpha$$

$$\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

soln Linear, Approx. Non-Linear: $\vec{y}(t) = \underbrace{k_1 e^{18t} \begin{pmatrix} 13 \\ -4 \end{pmatrix}}_{\text{dominates as } t \rightarrow \infty} + k_2 e^{-8t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$



(-2, 0):

$$\lambda_1 = -2 : (A - \lambda_1 I) \vec{v}_1 = \vec{0}$$

$$\begin{pmatrix} 0 & 4 \\ 0 & 8 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$4y_1 = 0$$

$$y_1 = 0, x_1 = \alpha$$

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\lambda_2 = 6 : (A - \lambda_2 I) \vec{v}_2 = \vec{0}$$

$$\begin{pmatrix} -8 & 4 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$-8x_2 + 4y_2 = 0$$

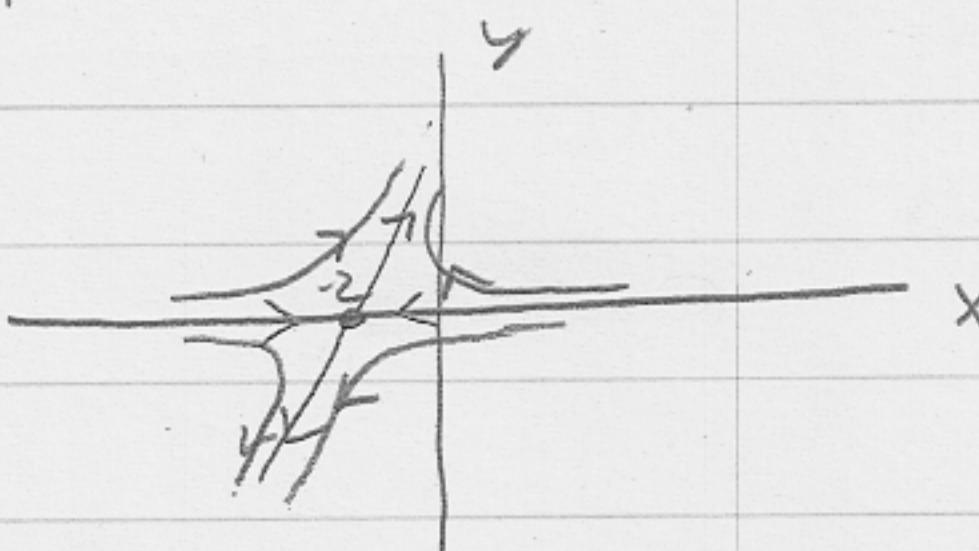
$$y_2 = 2x_2, x_2 = \alpha$$

$$y_2 = 2\alpha$$

$$\vec{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

soln Linear, Approx. Non-Linear: $\vec{y}(t) = k_1 e^{-2t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + k_2 e^{6t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

dominates as $t \rightarrow \infty$

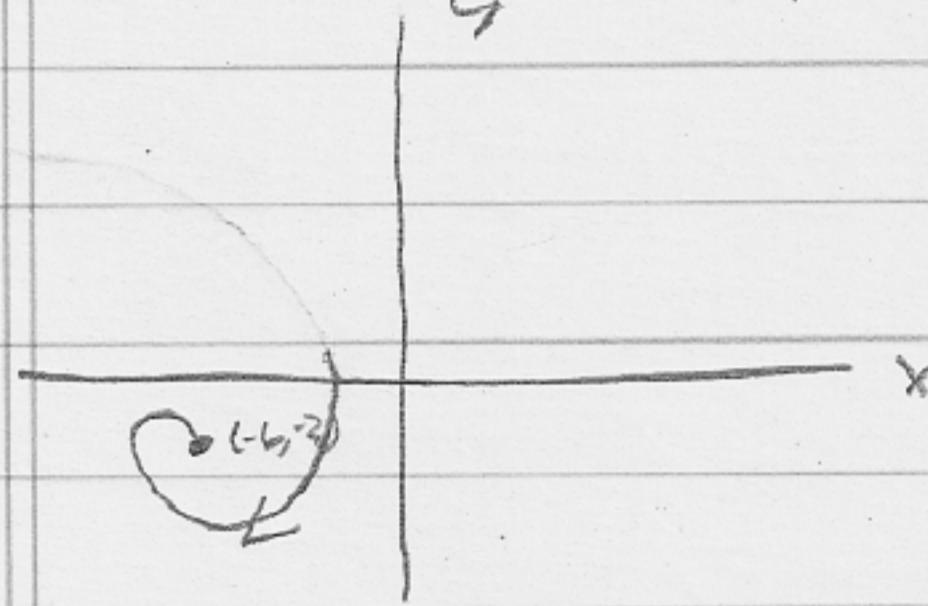


1. (c) (cont.)

$(-6, -2)$:

$$\lambda = -4 \pm 2\sqrt{5}i$$

spiral sink



$$\frac{d\vec{Y}}{dt} = A\vec{Y}$$

$$\frac{d\vec{Y}}{dt} = \begin{pmatrix} -6 & 2 \\ -2 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -6 \\ -2 \end{pmatrix}$$



2. Consider the same system from problem (1):

$$\frac{dx}{dt} = x^2 - 2xy + 2x = x(x - 2y + 2)$$

$$\frac{dy}{dt} = y^2 + xy + 8y = y(y + x + 8)$$

- a. On the entire xy-plane, sketch the x and y nullclines and indicate the direction of the vector field along each nullcline.

x-nullcline: $\frac{dx}{dt} = x(x - 2y + 2) = 0$

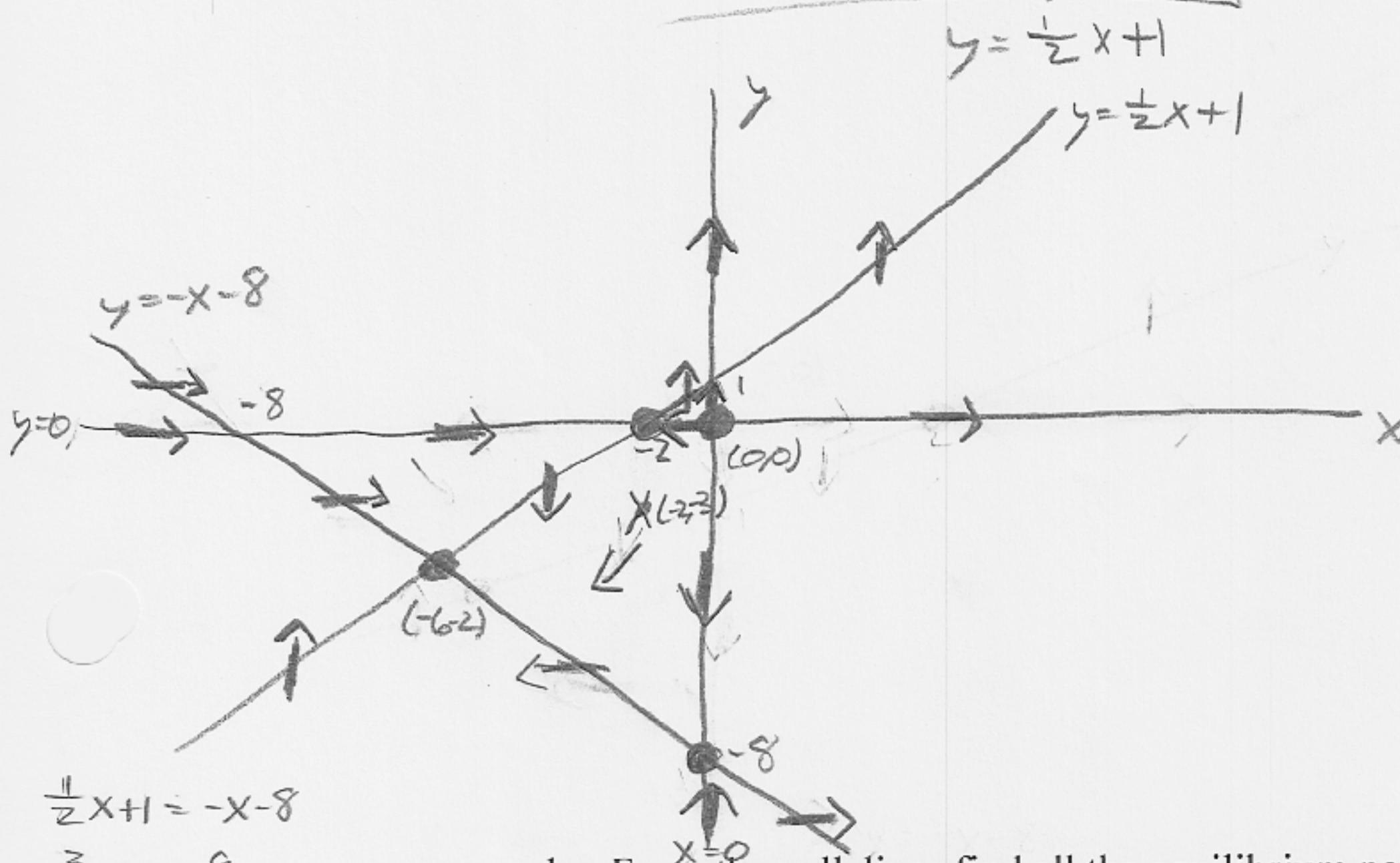
↑ or ↓

$$| \quad x=0, \quad x=2y-2$$

y-nullcline: $\frac{dy}{dt} = y(y + x + 8) = 0$

→ or ←

$$| \quad y=0 \quad y=-x-8$$



$$\frac{1}{2}x+1 = -x-8$$

$$\frac{3}{2}x = -9$$

$$x = -6$$

$$y = -(-6) - 8 = -2$$

- b. From the nullclines find all the equilibrium points

$$| \quad (0,0), (-2,0), (0,-8), \\ (-6,-2)$$

- c. What is the behavior of the solution with initial condition $\vec{Y}(0) = (-2, -3)$?

Solution will move towards EP at $(-6, -2)$.

x-nullcline:

$$(0, \frac{1}{2}): \frac{dx}{dt} = \frac{1}{4} + 4 > 0$$

$$(0, -\frac{1}{2}): \frac{dx}{dt} = \frac{1}{4} - 4 < 0$$

$$(0, -1): \frac{dx}{dt} = 81 + 0 - 72 > 0$$

$$(4, 3): \frac{dx}{dt} = 9 + 12 + 24 > 0$$

$$(-4, -1): \frac{dx}{dt} = 1 + 4 - 8 < 0$$

$$(-8, -3): \frac{dx}{dt} = 9 + 24 - 24 > 0$$

y-nullcline:

$$(-9, 0): \frac{dy}{dt} = 81 - 18 > 0$$

$$(-1, 0): \frac{dy}{dt} = 1 - 2 < 0$$

$$(1, 0): \frac{dy}{dt} = 1 + 2 > 0$$

$$(-9, 1): \frac{dy}{dt} = 81 - 18 - 18 > 0$$

$$(-2, -6): \frac{dy}{dt} = 4 - 24 - 4 < 0$$

$$(1, -9): \frac{dy}{dt} = 1 + 18 + 2 > 0$$

$$\frac{dx}{dt} = (x^2 - 2x)(x^2 + y^2 - 9)$$

3. Consider the system

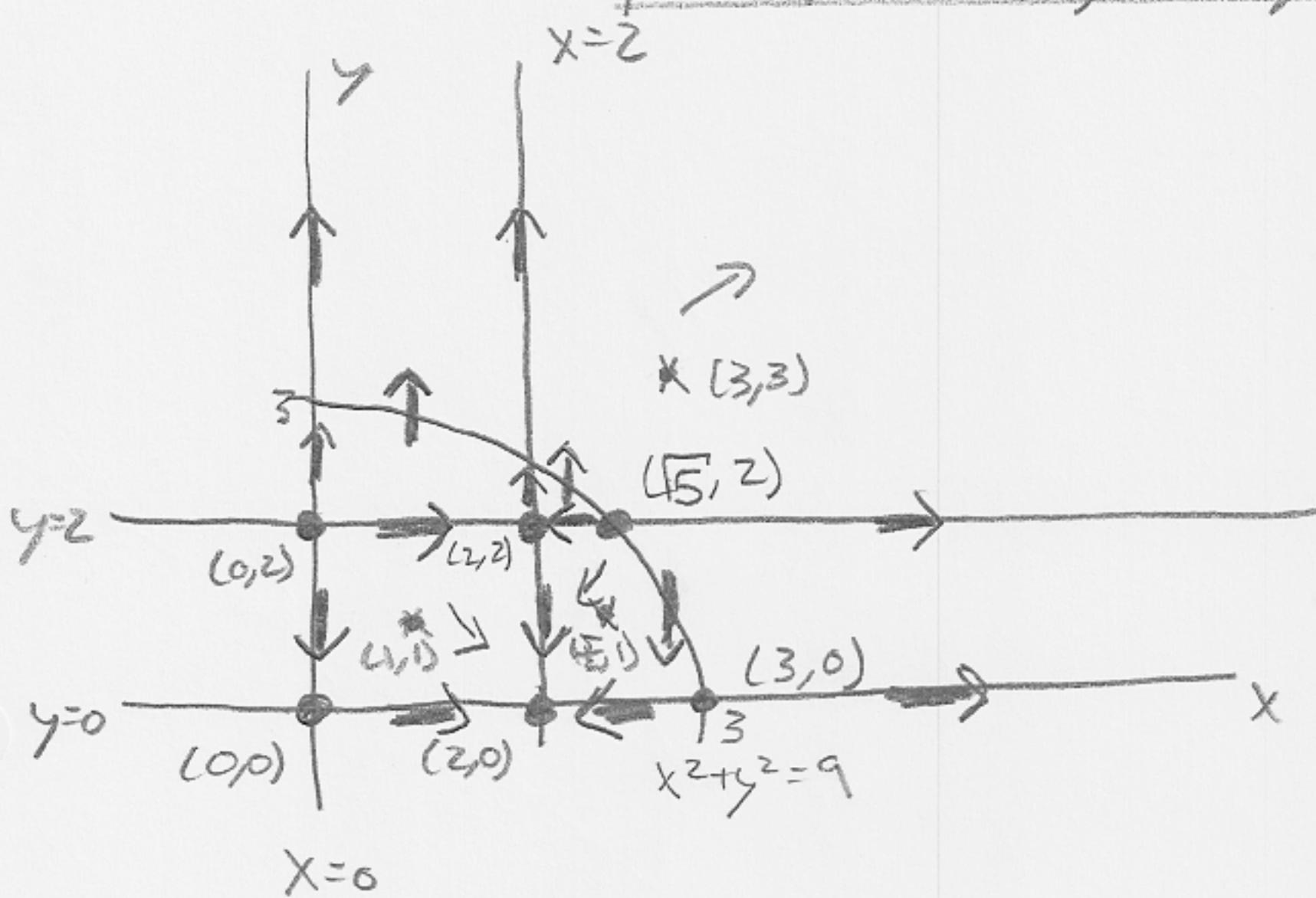
$$\frac{dy}{dt} = y^2 - 2y$$

- a. Sketch the x and y nullclines and indicate the direction of the vector field along each nullcline.. Restrict your graph to the first quadrant.

X-nullcline: $\frac{dy}{dt} = x(x-2)(x^2+y^2-9) = 0$

↑ or ↓

$$x=0, x=2, x^2+y^2=9$$



$$x^2 + y^2 = 9, y = \pm\sqrt{9-x^2}$$

$$z = \sqrt{9-x^2}$$

$$4 = 9 - x^2$$

$$x^2 = 5$$

$$x = \pm\sqrt{5}$$

$$x = \sqrt{5}, y = 2$$

- b. From the nullclines find the equilibrium points in the first quadrant.

$$(0,0), (0,2), (2,0), (2,2), (\sqrt{5}, 2), (3,0)$$

- c. What is the behavior of the solution with initial condition $\bar{Y}(0) = (1,1)$?

Solution will approach EP at $(2,0)$.

- d. What is the behavior of the solution with the initial condition

$\bar{Y}(0) = \left(\frac{5}{2}, 1\right)$? Solution will approach EP at $(2,0)$

- e. What is the behavior of the solution with the initial condition $\bar{Y}(0) = (3,3)$?

Solution will display infinite behavior.
 $x, y \rightarrow \infty$ as $t \rightarrow \infty$.

y-nullcline: $\frac{dy}{dt} = y(y-2) = 0$

↑ or ↓

$$y=0, y=2$$

X-nullcline

$$(0,1) : \frac{dy}{dt} = 1-2 < 0$$

$$(0,\frac{5}{2}) : \frac{dy}{dt} = \frac{25}{4} - \frac{10}{2} > 0$$

$$(2,1) : \frac{dy}{dt} = 1-2 < 0$$

$$(2,\frac{5}{2}) : \frac{dy}{dt} = \frac{25}{4} - \frac{10}{2} > 0$$

$$(1,\sqrt{5}) : \frac{dy}{dt} = 8 - 2\sqrt{5} > 0$$

$$(\sqrt{5}, \sqrt{5}) : \frac{dy}{dt} = 2 - 2\sqrt{2} < 0$$

y-nullcline

$$(1,0) : \frac{dx}{dt} = (1-2)(1-9) > 0$$

$$(\frac{5}{2}, 0) : \frac{dx}{dt} = (\frac{25}{4} - \frac{10}{2})(\frac{25}{4} - 9) < 0$$

$$(4,0) : \frac{dx}{dt} = (16-8)(16-9) > 0$$

$$(1,2) : \frac{dx}{dt} = (1-2)(1+4-9) < 0$$

$$(\frac{5}{2}, 2) : \frac{dx}{dt} = (\frac{25}{4} - \frac{10}{2})(\frac{25}{4} + 4 - 9) < 0$$

$$(4,2) : \frac{dx}{dt} = (16-8)(16+4-9) > 0$$