MATH348-Advanced Engineering Mathematics		**Homework: LA Part I**
LINEAR SYSTEMS : ALGEBRA, GEOMETRY, ROW-REDUCTION, DETERMINANTS, TRANSFORMATIONS		
Text: 7.1-7.3, 7.5, 7.7-7.8	Lecture Notes: N/A	Slides: N/A

Quote of Solutions to Homework: Linear Algebra Part I

Kesuke Miyagi: Ready?
Daniel LaRusso : Yeah, I guess so.
Kesuke Miyagi : Daniel-san, must talk. Walk on road. Walk right side, safe. Walk left side, safe. Walk middle, sooner or later, you get squished just like grape. Here, karate same thing. Either you karate do, yes, or karate do, no. You karate do, "guess so," just like grape. Understand?
Daniel LaRusso: Yeah, I understand.
Kesuke Miyagi: Ready?
Daniel LaRusso: Yeah, I'm ready.

Robert Mark Kamen : The Karate Kid (1984)

1. Matrix Multiplication

Define the *commutator* and *anti-commutator* of two square matrices to be,

$$[\cdot,\cdot]:\mathbb{C}^{n\times n}\times\mathbb{C}^{n\times n}\to\mathbb{C}^{n\times n},\ \text{such that}\ [\mathbf{A},\mathbf{B}]=\mathbf{AB}-\mathbf{BA},\quad \text{for all } \mathbf{A},\mathbf{B}\in\mathbb{C}^{n\times n},$$
$$\{\cdot,\cdot\}:\mathbb{C}^{n\times n}\times\mathbb{C}^{n\times n}\to\mathbb{C}^{n\times n},\ \text{such that}\ \{\mathbf{A},\mathbf{B}\}=\mathbf{AB}+\mathbf{BA},\quad \text{for all } \mathbf{A},\mathbf{B}\in\mathbb{C}^{n\times n},$$

respectively. Also define the *Kronecker delta* and *Levi-Civita* symbols to be,

$$\delta_{ij}:\mathbb{N}\times\mathbb{N}\to\{0,1\},\ \text{such that}\ \delta_{ij}=\begin{cases}1, & \text{if } i=j,\\ 0, & \text{if } i\neq j\end{cases}$$
$$\epsilon_{ijk}:(i,j,k)\to\{-1,0,1\},\ \text{such that}\ \epsilon_{ijk}=\begin{cases}1, & \text{if } (i,j,k) \text{ is } (1,2,3),(2,3,1) \text{ or } (3,1,2),\\ -1, & \text{if } (i,j,k) \text{ is } (3,2,1),(1,3,2) \text{ or } (2,1,3),\\ 0, & \text{if } i=j \text{ or } j=k \text{ or } k=i\end{cases}$$

respectively. Also define the so-called *Pauli spin-matrices* (PSM) to be,

$$\sigma_1=\sigma_x=\begin{bmatrix}0&1\\1&0\end{bmatrix},\quad \sigma_2=\sigma_y=\begin{bmatrix}0&-i\\i&0\end{bmatrix},\quad \sigma_3=\sigma_z=\begin{bmatrix}1&0\\0&-1\end{bmatrix}.$$

1.1. **The PSM are *self-adjoint* matrices.** Show that $\sigma_m=\sigma_m^{\text{H}}$ for $m=1,2,3$.

A matrix is self-adjoint if it is equal to its own complex conjugated transpose. That is, $\mathbf{A}$ is self-adjoint (also called Hermitian) if $\mathbf{A}=\mathbf{A}^{\text{H}}=\bar{\mathbf{A}}^{\text{T}}$. Notice that if a matrix has only real entries then self-ajoint implies symmetric. Clearly, for $m=1$ and $m=3$ the matrix is real and symmetric an therefore self-adjoint. When $m=2$ we write,

$$\begin{aligned}\sigma_y^{\text{H}}&=\bar{\sigma_y}^{\text{T}}\\ &=\overline{\begin{bmatrix}0&-i\\i&0\end{bmatrix}}^{\text{T}}\\ &=\overline{\begin{bmatrix}0&i\\-i&0\end{bmatrix}}\\ &=\begin{bmatrix}0&\bar{i}\\\bar{i}&0\end{bmatrix}\\ &=\sigma_y.\end{aligned}$$

Hence σ_y is self-adjoint.

1.2. **The PSM are *unitary* matrices.** Show that $\sigma_m^2 = \mathbf{I}$ for $m = 1, 2, 3$ where $[\mathbf{I}]_{ij} = \delta_{ij}$.

A matrix is unitary if $\mathbf{U}^{\text{H}}\mathbf{U} = \mathbf{I}$. That is, a matrix is unitary if its adjoint is also its own inverse. Since we already know that the PSM are self-adjoint, $\sigma_m^{\text{H}} = \sigma_m$, we need only show that $\sigma_m^2 = \mathbf{I}$ for $m = 1, 2, 3$.

$$
\begin{aligned}
\sigma_1^2 &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\
&\begin{bmatrix} 0 \cdot 0 + 1 \cdot 1 & 0 \cdot 1 + 1 \cdot 0 \\ 1 \cdot 0 + 0 \cdot 1 & 1 \cdot 1 + 0 \cdot 0 \end{bmatrix} \\
&= \mathbf{I} \\
\sigma_2^2 &= \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \\
&= \mathbf{I} \\
\sigma_3^2 &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\
&= \mathbf{I}
\end{aligned}
$$

Hence, the PSM are unitary matrices.

1.3. **Trace and Determinant.** Show that $\text{tr}(\sigma_m) = 0$ and $\det(\sigma_m) = -1$ for $m = 1, 2, 3$. Given a matrix,

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

we define the trace and determinant of $\mathbf{A}$ with the following matrix functions,

$$T = \text{tr}(\mathbf{A}) = a + d, \qquad D = \det(\mathbf{A}) = ad - bc.$$

It is easy to see that the PSM are traceless matrices. That is $\text{tr}(\sigma_m) = 0 + 0$ for $m = 1, 2$ and $\text{tr}(\sigma) = 1 - 1 = 0$ for $m = 3$. Another quick check shows,

$$
\begin{aligned}
\det(\sigma_1) &= 0 \cdot 0 - 1 \cdot 1 = -1 \\
\det(\sigma_2) &= 0 \cdot 0 - i(-i) = -1 \\
\det(\sigma_3) &= 1 \cdot (-1) - 0 \cdot 0 = -1
\end{aligned}
$$

Generally, one can show that unitary matrices have determinant one or negative one.

1.4. **Anti-Commutation Relations.** Show that $\{\sigma_i, \sigma_j\} = 2\delta_{ij}\mathbf{I}$ for $i = 1, 2, 3$ and $j = 1, 2, 3$.
If we notice a couple of identities first then the busy-work is reduced. Specifically, note that $\{\mathbf{A}, \mathbf{B}\} = \{\mathbf{B}, \mathbf{A}\}$ and $\{\mathbf{A}, \mathbf{A}\} = 2\mathbf{A}^2$. So, by the self-adjoint and unitary properties we have, $\{\sigma_i, \sigma_i\} = 2\sigma_i^2 = 2\mathbf{I}$ for $i = 1, 2, 3$. For all other cases we expect the anti-commutator to

return a zero matrix. To show this we need only consider the following ordered pairs, $(i, j) = \{(1, 2), (1, 3), (2, 3)\}$. Doing so gives,

$$
\begin{aligned}
\{\sigma_1, \sigma_2\} &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} + \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} + \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} \\
&= \mathbf{0} \\
\{\sigma_1, \sigma_3\} &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \\
&= \mathbf{0} \\
\{\sigma_2, \sigma_3\} &= \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} + \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix} \\
&= \mathbf{0}
\end{aligned}
$$

Taken together these statements imply that $\{\sigma_i, \sigma_j\} = 2\delta_{ij}$ for $i = 1, 2, 3$ and $j = 1, 2, 3$.

1.5. **Commutation Relations.** Show that $[\sigma_i, \sigma_j] = 2\sqrt{-1}\sum_{k=1}^{3} \epsilon_{ijk}\sigma_k$ for $i = 1, 2, 3$ and $j = 1, 2, 3$.

Again, to make quick work of this we notice that $[\mathbf{A}, \mathbf{B}] = -[\mathbf{B}, \mathbf{A}]$ and $[\mathbf{A}, \mathbf{A}] = \mathbf{0}$. This implies that if the subscripts are the same then $[\sigma_i, \sigma_i] = 2\sqrt{-1}\sum_{k=1}^{3} \epsilon_{iik}\sigma_k = 0$ since $\epsilon_{iik} = 0$ by definition. As before we now only need to determine the commutator relation for $(i, j) = \{(1, 2), (1, 3), (2, 3)\}$ and note that if the subscripts are switched then a negative sign is introduced. Moreover, we can use the previous results since the commutator is just the anti-commutator with a subtraction. Thus,

$$
[\sigma_1, \sigma_2] = 2\sqrt{-1}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = 2\sqrt{-1}\sigma_3
$$

$$
\begin{aligned}
[\sigma_1, \sigma_3] &= 2\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \\
&= -2\sqrt{-1}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = -2\sqrt{-1}\sigma_2
\end{aligned}
$$

$$
[\sigma_2, \sigma_3] = 2\sqrt{-1}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = 2\sqrt{-1}\sigma_1
$$

this and noting the anti-symmetry of the commutator establishes the following pattern:

i	j	$\sum_{k=1}^{3} \epsilon_{ijk}\sigma_k$
1	2	$\epsilon_{121}\sigma_1 + \epsilon_{122}\sigma_2 + \epsilon_{123}\sigma_3 = \sigma_3$
1	3	$\epsilon_{131}\sigma_1 + \epsilon_{132}\sigma_2 + \epsilon_{133}\sigma_3 = -\sigma_2$
2	3	$\epsilon_{231}\sigma_1 + \epsilon_{232}\sigma_2 + \epsilon_{233}\sigma_3 = \sigma_1$

Noting that $\epsilon_{ijk} = -\epsilon_{jik}$ completes the pattern.

2. Solutions Sets to Linear Systems of Algebraic Equations

Given,

$$\mathbf{A}_1 = \begin{bmatrix} 1 & -3 & 0 \\ -1 & 1 & 5 \\ 0 & 1 & 1 \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} 6 & 18 & -4 \\ -1 & -3 & 8 \\ 5 & 15 & -9 \end{bmatrix}, \quad \mathbf{A}_3 = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 3 \end{bmatrix}, \quad \mathbf{A}_4 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}, \quad \mathbf{A}_5 = \begin{bmatrix} 5 & 3 \\ -4 & 7 \\ 9 & -2 \end{bmatrix},$$

$$\mathbf{b}_1 = \begin{bmatrix} 5 \\ 2 \\ 0 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} 20 \\ 4 \\ 11 \end{bmatrix}, \quad \mathbf{b}_3 = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{b}_4 = \begin{bmatrix} 10 \\ 20 \\ 30 \end{bmatrix}, \quad \mathbf{b}_5 = \begin{bmatrix} 22 \\ 20 \\ 15 \end{bmatrix}.$$

2.1. **Algebra.** Find all solutions to $\mathbf{A}_i\mathbf{x} = \mathbf{b}_i$ for $i = 1, 2, 3, 4, 5$. The following row-equivalences can be checked via *computational tools*.[1]

$$[\mathbf{A}_1 \,|\mathbf{b}_1] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right] \tag{1}$$

$$[\mathbf{A}_2 \,|\mathbf{b}_2] \sim \left[\begin{array}{ccc|c} 1 & 3 & 0 & 4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \tag{2}$$

$$[\mathbf{A}_3 \,|\mathbf{b}_3] \sim \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \tag{3}$$

$$[\mathbf{A}_4 \,|\mathbf{b}_4] \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 10 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \tag{4}$$

$$[\mathbf{A}_5 \,|\mathbf{b}_5] \sim \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \tag{5}$$

Since the reduced row echelon matrices share the same solutions as the original linear systems we have,

$$(1) \implies \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix},$$

$$(2) \implies \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 - 3x_2 \\ x_2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 - 3t \\ t \\ 1 \end{bmatrix}, \quad t \in \mathbb{R},$$

$$(4) \implies \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10 - 2x_2 - 3x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10 - 2t - 3s \\ t \\ s \end{bmatrix}, \quad t, s \in \mathbb{R}.$$

Since in systems three and five the reduced row echelon forms have an inconsistent row their associated systems do not have solutions. More precisely, the first two equations of both systems have points of common intersection but the third equation does not share these points.

2.2. **Geometry.** Describe or plot the geometry formed by the linear systems and their solution sets.

[1] These calculations should be done by hand. There is no replacement for this type of practice. Computational tools should be used to check your work either as you go or at the end of your hand-calculations. A good tool can be found through the external material links on the ticc website. The tool-kit will row-reduce a matrix and show you the steps it used.

- System one is the algebraic representation of three planes in space, which share a common point of intersection, $(2, -1, 1)$.

- System two is the algebraic representation of three planes in space, which share common points of intersection. There are infinitely many of these points defined by $\mathbf{x} = [4 - 3t \; t \; 1]^{\mathrm{T}}$, which parameterizes a line in space.

- System three is the algebraic representation of three planes in space, which share no common points of intersection. This does not mean that the planes do not intersect one another. It just means that they do not do so simultaneously.

- System four is the algebraic representation of three planes in space, which share common points of intersection. There are infinitely many of these points defined by $\mathbf{x} = [10 - 2t - 3s \; t \; s]^{\mathrm{T}}$, which parameterizes a plane in space.

- System five is the algebraic representation of three lines in space, which share no common points of intersection. Again, these lines intersect one another but do not have any points in space where they do so simultaneously.

3. Square Coefficient Data and Matrix Inversion

Given,

$$\mathbf{A} = \begin{bmatrix} 3 & 6 & 7 \\ 0 & 2 & 1 \\ 2 & 3 & 4 \end{bmatrix}.$$

3.1. **Matrix Inverse: Take One.** Find $\mathbf{A}^{-1}$ using the Gauss-Jordan Method. (pg.317)

When asked to calculate an inverse matrix this is the algorithm to use. It is simpler and less computationally intensive than other methods and is roughly what a computational device does when asked to find an inverse matrix.

$$\left[\begin{array}{ccc|ccc} 3 & 6 & 7 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 2 & 3 & 4 & 0 & 0 & 1 \end{array}\right] \begin{array}{c} R1 \to R1 - 3R2 \\ \sim \\ R3 \to 2R1 - 3R3 \end{array} \left[\begin{array}{ccc|ccc} 3 & 0 & 4 & 1 & -3 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & 3 & 2 & 2 & 0 & -3 \end{array}\right] \begin{array}{c} \\ \sim \\ 2R3 - 3R2 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 3 & 0 & 4 & 1 & -3 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 4 & -3 & -6 \end{array}\right] \begin{array}{c} R1 \to R1 - 4R3 \\ \sim \\ R2 \to R2 - R3 \end{array} \left[\begin{array}{ccc|ccc} 3 & 0 & 0 & -15 & 9 & 24 \\ 0 & 2 & 0 & -4 & 4 & 6 \\ 0 & 0 & 1 & 4 & -3 & -6 \end{array}\right] \begin{array}{c} R1 \to R1/3 \\ \sim \\ R2 \to R2/2 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -5 & 3 & 8 \\ 0 & 1 & 0 & -2 & 2 & 3 \\ 0 & 0 & 1 & 4 & -3 & -6 \end{array}\right] \implies A^{-1} = \begin{bmatrix} -5 & 3 & 8 \\ -2 & 2 & 3 \\ 4 & -3 & -6 \end{bmatrix}$$

3.2. **Matrix Inverse: Take Two.** Find $\mathbf{A}^{-1}$ using the cofactor representation. (Theorem 2 pg.318)

There are, of course, other ways to find $\mathbf{A}^{-1}$. The following method uses determinants and provides a general representation of an inverse matrix, if it exists. First we must find $\det(\mathbf{A})$. Using the cofactor expansion of the determinant we have,

$$\begin{aligned} \det(\mathbf{A}) &= 3\det\begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix} - 0 \cdot \det\begin{pmatrix} 6 & 7 \\ 3 & 4 \end{pmatrix} + 2\det\begin{pmatrix} 6 & 7 \\ 2 & 1 \end{pmatrix} \\ &= 3(5) - 0(3) + 2(-8) = 15 - 16 = -1 \end{aligned}$$

Using the cofactor formula we have,

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})} \begin{bmatrix} c_{11} & c_{21} & c_{31} \\ c_{12} & c_{22} & c_{32} \\ c_{13} & c_{23} & c_{33} \end{bmatrix} = \frac{1}{-1} \begin{bmatrix} 5 & -3 & -8 \\ 2 & -2 & -3 \\ -4 & 3 & 6 \end{bmatrix} = \begin{bmatrix} -5 & 3 & 8 \\ -2 & 2 & 3 \\ 4 & -3 & -6 \end{bmatrix},$$

where $c_{ij} = (-1)^{i+j} \det(\mathbf{A}_{ij})$. Since, this method requires the use of determinants it is computationally intensive, but does highlight the connection between $\det(\mathbf{A}) = 0$ and non-invertibility. A typical use of this method is to study how elements of $\mathbf{A}^{-1}$ changes with changes to $\mathbf{A}$.

3.3. **Check Step.** Verify that this inverse matrix is correct.

It is easy to verify that we have found the correct matrix inverse of $\mathbf{A}$. We have found $\mathbf{A}^{-1}$ using two different methods and gotten the same answer but if we are still worried then we conduct the following matrix multiplication $\mathbf{AA}^{-1} = \mathbf{I}$. Doing so gives,

$$\mathbf{AA}^{-1} = \begin{bmatrix} 3 & 6 & 7 \\ 0 & 2 & 1 \\ 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} -5 & 3 & 8 \\ -2 & 2 & 3 \\ 4 & -3 & -6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{I},$$

which implies that $\mathbf{A}^{-1}$ is the inverse of $\mathbf{A}$.

3.4. **Solutions to Linear Systems.** Using $\mathbf{A}^{-1}$ find the unique solution to $\mathbf{Ax} = \mathbf{b} = [b_1\, b_2\, b_3]^{\mathrm{T}}$.

Since there is an inverse matrix for $\mathbf{A}$ there must exist a unique solution regardless of the choice of $\mathbf{b} \in \mathbb{R}^3$. Algebraically we have, $\mathbf{Ax} = \mathbf{b} \iff \mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$, where

$$\begin{aligned} \mathbf{x} &= \mathbf{A}^{-1}\mathbf{b} \\ &= \begin{bmatrix} -5 & 3 & 8 \\ -2 & 2 & 3 \\ 4 & -3 & -6 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \\ &= \begin{bmatrix} -5b_1 + 3b_2 + 8b_3 \\ -2b_1 + 2b_2 + 3b_3 \\ 4b_1 - 3b_2 - 6b_3 \end{bmatrix}. \end{aligned}$$

4. Determinants

Given,

$$\mathbf{A} = \begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix}.$$

4.1. ***Vandermonde Determinant.*** Show that the $\det(\mathbf{A}) = (c-a)(c-b)(b-a)$.

This calculation is easiest done in conjunction with row-reduction. The following row-reduction,

$$\begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix} \begin{array}{c} R2 \to R2 - R1 \\ \sim \\ R3 \to R3 - R1 \end{array} \begin{bmatrix} 1 & a & a^2 \\ 0 & b-a & b^2 - a^2 \\ 0 & c-a & c^2 - a^2 \end{bmatrix} \begin{array}{c} R3 \to R3 - \dfrac{(c-a)}{(b-a)} R2 \\ \sim \end{array} \begin{bmatrix} 1 & a & a^2 \\ 0 & (b-a) & b^2 \\ 0 & 0 & c^2 - a^2 - (c-a)(b^2-a^2)/(b-a) \end{bmatrix},$$

implies that,

$$\begin{aligned} \det(\mathbf{A}) &= 1 \cdot (b-a) \cdot \left(c^2 - a^2 - (c-a)\frac{b^2 - a^2}{b-a} \right) \\ &= (b-a)\left((c-a)(c+a) - (c-a)\frac{(b-a)(b+a)}{b-a} \right) \\ &= (b-a)(c-a)\,(c+a-b+a) \\ &= (b-a)(c-a)\,(c-b) \end{aligned}$$

4.2. **Application.** Determine which of the following sets of points can be uniquely interpolated by the polynomial $p(t) = a_0 + a_1 t + a_2 t^2$.

$$\begin{aligned} S_1 &= \{(1,12), (2,15), (3,16)\} \\ S_2 &= \{(1,12), (1,15), (3,16)\} \\ S_3 &= \{(1,12), (2,15), (2,15)\} \end{aligned}$$

First notice that even though the polynomial equation is nonlinear in the t-variable it is linear in the coefficients and $p(t) = \begin{bmatrix} 1 \; t \; t^2 \end{bmatrix}^{\mathrm{T}} \cdot [a_0\; a_1\; a_2]^{\mathrm{T}}$. Every point, $(t_i, p_i = p(t_i))$, given defines a new polynomial and in the case of three points we have the following linear system,

$$\begin{bmatrix} 1 & t_0 & t_0^2 \\ 1 & t_1 & t_1^2 \\ 1 & t_2 & t_2^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} p_0 \\ p_1 \\ p_2 \end{bmatrix}.$$

Using the previous result we now have the interpretation that if any of the time values are the same then the determinant of the coefficient matrix is zero and if a solution exists to the interpolation problem then this solution is NOT unique. Thus, of the three sets of points only the set S_1 admits a unique interpolating polynomial.

For completion we can apply some common sense to the remaining sets. Looking at S_2 we see that the graph must pass through $(1, 12)$ and $(1, 15)$. The vertical line test tells us no function can do this. Looking at S_3 we see that the second and third point are the same. A quick reduction shows that there is a free-variable in the augmented matrix. This implies that while there is only one line that connects two points in space there are many quadratic polynomials that connect two points in space.

5. Rotation Transformations in $\mathbb{R}^2$ and $\mathbb{R}^3$

Given,

$$\mathbf{A}(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}.$$

5.1. **The Unit Circle.** Show that the transformation $\mathbf{A}\hat{\mathbf{i}}$ rotates $\hat{\mathbf{i}} = [1\ 0]^{\mathrm{T}}$ counter-clockwise by an angle θ and defines a parametrization of the *unit circle*. What matrix would undo this transformation?

The affect of the transformation applied to $\hat{\mathbf{i}}$ is given by,

$$\mathbf{A}\hat{\mathbf{i}} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix},$$

which is a counterclockwise parameterization of the unit-circle for $\theta \in [0, 2\pi)$. To undo this consider the matrix $\mathbf{A}^{\mathrm{T}}$ to get $\mathbf{A}^{\mathrm{T}}\hat{\mathbf{i}} = [\cos(\theta)\ \ -\sin(\theta)]^{\mathrm{T}}$. This leads us to conclude that $\mathbf{A}\mathbf{A}^{\mathrm{T}} = \mathbf{I}$, which means that $\mathbf{A}$ is a orthogonal transformation or a ridged change of coordinates.

5.2. **Determinant.** Show that $\det(\mathbf{A}) = 1$.
$\det(\mathbf{A}) = \cos(\theta)\cos(\theta) - -\sin(\theta)\sin(\theta) = 1$

5.3. **Orthogonality.** Show that $\mathbf{A}^{\mathrm{T}}\mathbf{A} = \mathbf{A}\mathbf{A}^{\mathrm{T}} = \mathbf{I}$.

To formally show the orthogonality we verify either of the previous multiplications.

$$\mathbf{A}\mathbf{A}^{\mathrm{T}} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} = \begin{bmatrix} \cos^2(\theta) + \sin^2(\theta) & \cos(\theta)\sin(\theta) - \cos(\theta)\sin(\theta) \\ \cos(\theta)\sin(\theta) - \cos(\theta)\sin(\theta) & \cos^2(\theta) + \sin^2(\theta) \end{bmatrix} = \mathbf{I}$$

5.4. **Rotations in $\mathbb{R}^3$.** Given,

$$\mathbf{R}_1(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}, \qquad \mathbf{R}_2(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \qquad \mathbf{R}_3(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Describe the transformations defined by each of these matrices on vectors in $\mathbb{R}^3$.

It is best to think about this in terms of linear combinations of columns. Consider,

$$\mathbf{R}_1(\theta)\mathbf{x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ \cos(\theta) \\ \sin(\theta) \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ -\sin(\theta) \\ \cos(\theta) \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \mathbf{A} \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} \end{bmatrix},$$

which implies that $\mathbf{R}_1$ leaves the x_1 component of the vector unchanged and rotates the x_2, x_3 component of the vector in the x_2, x_3-plane. Similar arguments show that $\mathbf{R}_2$ leaves x_2 unchanged and rotates the vector in the x_1, x_3-plane while $\mathbf{R}_3$ leave x_3 and rotates in the x_1, x_2-plane. This lays the ground-work for the so-called *Euler angles*, which provides a systematic way to rotates geometries in $\mathbb{R}^3$. An interesting consequence of the relationship between rotations in $\mathbb{R}^3$ and matrix algebra is that since these matrices do not commute, $[\mathbf{R}_1, \mathbf{R}_2] \neq \mathbf{0}$ for $\theta \neq 0$, the order one conducts the rotations matter. [2]

[2]To see this take a textbook and orient it so that the spine is facing you and the cover is facing up. Rotate the text clockwise $\pi/2$ about the z-axis then rotate it $\pi/2$ about the y-axis. At this point I am looking at the back of the textbook and the text is upside-down. Now do the rotations in reverse order and you will see the lack of commutivity.