Diffraction

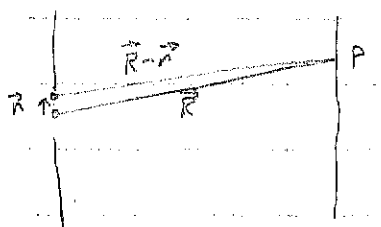
intuitive concept: Huygens point sources across aperture
- needs refinement

scalar spherical wave: $E(r) = E_0 \dfrac{e^{ikr}}{r}$
- E_0 = complex (e.g. $Ae^{i\phi}$)
- take real part for real field.
- suppressing time dependence.
- note intensity ~ $E^2 \sim 1/r^2$

implement intuitive idea: sum all sources across aperture.
let $\Psi_{tr}(\vec{r})$ be scalar field in aperture plane $(x, y, 0)$
- simplest form ~ rect( )

diffracted field is the convolution of spherical wave with input aperture wave:

$$\Psi_{dif}(\vec{R}) \propto \int \Psi_{tr}(\vec{r}) \frac{e^{ik|\vec{R}-\vec{r}|}}{|\vec{R}-\vec{r}|}\, dS$$



- fixed point $\vec{R}$ results from sum across aperture
- note phase changes are impt: $2\pi \frac{|\vec{R}-\vec{r}|}{\lambda}$

here $\Psi_{tr}(\vec{r}) = \text{rect}(x/a)\,\text{rect}(y/a)$ as example.

Is this legitimate? Yes, but...

Green's fcn analysis $\rightarrow$ $\Psi_{diff}(\vec{R}) = \dfrac{1}{i\lambda}\displaystyle\int dS\, \Psi_{tr}(\vec{r}) \frac{e^{ik|\vec{R}-\vec{r}|}}{|\vec{R}-\vec{r}|} \frac{(1+\cos\theta)}{2}$

$\frac{1}{i}$ = phase shift $\quad \frac{1}{\lambda}$ for units

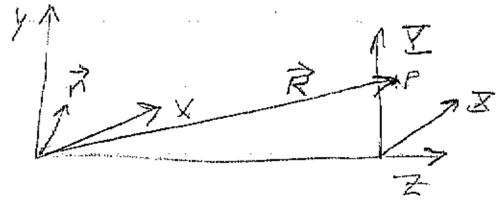
$\frac{1}{2}(1+\cos\theta)$ = "obliquity factor" $\simeq 1$ to make sure light $\rightarrow$ forward.

Fresnel + Fraunhofer diffraction

Starting point:

$$\psi_{dif}(\vec{R}) = \frac{1}{i\lambda}\int \psi_{inc}(\vec{r})\,\frac{e^{ik|\vec{R}-\vec{r}|}}{|\vec{R}-\vec{r}|}\,dx\,dy$$

this integral advances the field from one plane ($\vec{r}$, or x, y) to another plane ($\vec{R}$, X, Y)



Comment: it is helpful to imagine an aperture
- there may be one
- in general, this integral will work as long as most energy is propagating in the forward direction.

Try to do integral

$$|\vec{R}-\vec{r}| = \sqrt{(X-x)^2+(Y-y)^2+z^2}$$

general idea: $x, y \ll R$

zeroth order $|\vec{R}-\vec{r}| \to R$ ok in denominator.

for phase, expand

pull out $R = \sqrt{X^2+Y^2+z^2}$

$$|\vec{R}-\vec{r}| = R\left[\frac{X^2+Y^2+z^2-2Xx-2Yy+x^2+y^2}{R^2}\right]^{1/2}$$

$$= R\left[1+\frac{x^2+y^2}{R^2}-\frac{2(Xx+Yy)}{R^2}\right]^{1/2}$$

we want to approximate $\sqrt{\ }$

require $x^2+y^2 \approx a^2$ (aperture or beam size)

$$\ll R^2$$

and $(X x + Y y)/R^2 \ll 1$

now $k|\vec{R}-\vec{r}| \approx kR\left(1 + \frac{x^2+y^2}{2R} - \frac{Xx+Yy}{R^2}\right)$

put it together

$$\psi_{dif}(X,Y) = \frac{1}{i\lambda}\int \psi_{inc}(x,y)\,\frac{e^{ikR}}{R}\,e^{i\frac{k}{2R}(x^2+y^2) - ik\frac{(Xx+Yy)}{R}}\,dx\,dy$$

let $a = \sqrt{x_{max}^2 + y_{max}^2}$ be extent of aperture/beam

if $\frac{ka^2}{2R} = \frac{\pi a^2}{\lambda}\cdot\frac{1}{R} \ll 1$ then drop this term

→ Fraunhofer diffraction.

Otherwise keep it: Fresnel diffraction

Fraunhofer:

$$\psi_{dif}(X,Y) = \frac{e^{ikR}}{i\lambda R}\iint \psi_{tr}(x,y)\,e^{-i\left(\frac{kX}{R}x + \frac{kY}{R}y\right)}\,dx\,dy$$

↳ transmitted field.

this is like a Fourier transform

- spatial frequency variables $\beta_x = kX/R$
  $\beta_y = kY/R$
- the far-field pattern is angular $k_x = k\sin\theta_x$
  $k_y = k\sin\theta_y$
- physics has given us a different sign.

$$F(\beta_x,\beta_y) = \tilde{\mathcal{F}}_{xy}\{f(x,y)\} = \iint f(x,y)\,e^{-i\beta_x x}\,e^{-i\beta_y y}\,dx\,dy$$

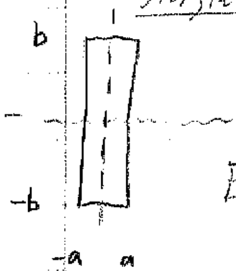
Fraunhofer examples

$$E_{dif}(X,Y) = \frac{e^{ikR}}{i\lambda R} \iint E_{inc}(x,y)\, A(x,y)\, e^{-i(\beta_x x + \beta_y y)}\, dx\, dy$$

$$= \frac{e^{ikR}}{i\lambda R} \mathfrak{F}\{E_{inc}(x,y)\, A(x,y)\}$$

$$\beta_x = +kX/R = +k \sin\theta_x \qquad \beta_y = +kY/R = -k\sin\theta_y$$

Single slit : $A(x,y) = \text{rect}(x/2a)\, \text{rect}(y/2b)$



$E_{inc}(x,y) = E_0$ (even illumination)

$$E_{dif}(X,Y) = \frac{E_0 e^{ikR}}{i\lambda R} \mathfrak{F}_x\{\text{rect}(x/2a)\}\, \mathfrak{F}_y\{\text{rect}(y/2b)\}$$

$$= \frac{E_0 e^{ikR}}{i\lambda R} 4ab \,\text{sinc}(\beta_x a)\, \text{sinc}(\beta_y b)$$

intensity $I = \frac{1}{2}\varepsilon_0 c n |E|^2$

$$I_{dif} = \underbrace{\frac{1}{2}\varepsilon_0 c n E_0^2}_{I_0} \frac{16 a^2 b^2}{\lambda^2 R^2} \text{sinc}^2\left(ka \sin\theta_x\right) \text{sinc}^2\left(kb \sin\theta_y\right)$$

1st zero at $ka \sin\theta_x = \pi$ or $\underline{\sin\theta_x = (\lambda/2a)\frac{1}{n}}$

- inside medium $n > 1$ diffn. angles are smaller.
- observing intensity loses phase information.
  - > where $E < 0$ $I > 0$

small angles : $\underline{\theta_x \doteq \frac{\lambda}{2a}}$