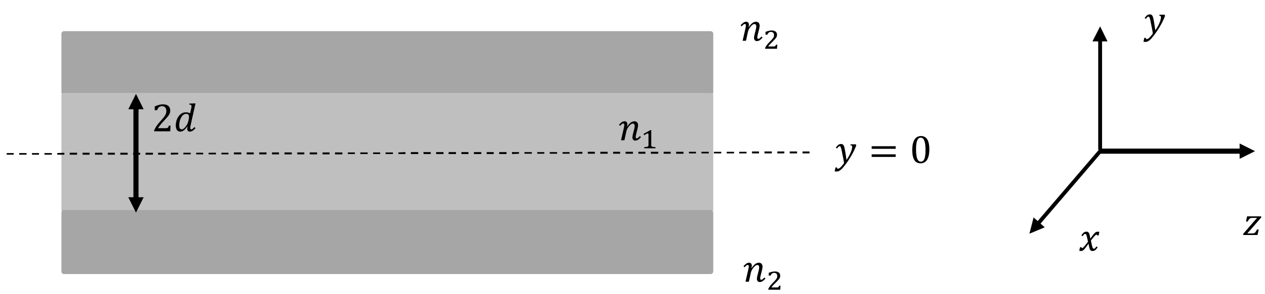
PHGN 462 Homework 9

1) (From Pollack and Stump 14.1)

For the TE(n) mode of propagation in the space between parallel conducting planes, show that $\vec{S}_{avg} = u_{avg} v_{gr} \hat{z}$, where u is the field energy density. Here the subscript avg implies an average over both t and the y coordinate. Comment on whether this helps us figure out which velocity (phase or group) is the velocity at which the mode "actually" propagates.

2) The waveguides that we're dealing with in class all have conducting boundaries, but we have the tools we need to consider waveguides made of dielectric materials, too. These are arguably the most important kind of modern waveguide, since that category includes optical fibers. We're going to construct TE solutions for a waveguide made up of a planar slab of dielectric with two other dielectrics above and below it. The top and bottom layers are of index n_2 and go up and down effectively forever. The inner layer is of index n_1 and is of thickness $2d$. We'll let the dead center be $y = 0$.



An optical fiber is basically this, but with a cylindrical geometry. The inner and outer layers are referred to as the core and cladding, respectively. Note that in general we have $n_1 > n_2$. The ray optics view of this system is that the light propagates down the center, totally internally reflecting off of the two interfaces – and total internal reflection requires that you be going from high index into low index.

Anyway, this is an awful lot like a parallel plate waveguide (which we know how to deal with), except that the top and bottom regions aren't conducting, so the fields in those regions aren't forced to be exactly zero. However, if the radiation is bouncing along by way of total internal reflection, the fields in the top and bottom regions are quite likely to be evanescent (which we also know how to deal with).

Our starting points include the basic form of an evanescent E-field and the basic form of TE modes in a parallel-plate conductor:

$$\vec{E}_{evan} = E_0 e^{-k_y y} e^{i(k_z z - \omega t)} \hat{\imath} \qquad \vec{E}_{TE} = E_0 \sin k_y y \, e^{i(k_z z - \omega t)} \hat{\imath}$$

We're going to use these as trial functions to build up solutions for the dielectric slab waveguide.

a) Write the trial E-fields for each of the top, middle, and bottom regions. You'll have to slightly adapt the evanescent field equation above to account for the fact that the evanescent fields begin at $y = \pm d$ and need to decay as we go farther up/down. Then use Faraday's law to find the corresponding B-fields in those regions. Make no assumptions beyond the basic form of the fields. In other words, don't assume that the k's, E_0's, or ω's in different regions are the same.

b) Apply our four basic boundary conditions on E and B to these fields. Figure out which of the k's, E_0's, or ω's are the same in each region, and show that k_y and k'_y are related by:

$$-\cot k_y d = \frac{k'_y}{k_y}$$

Where k_y and k'_y are the y components of the wavevector in media 1 and 2, respectively.

c) Use the fact that the E-fields in regions 1 and 2 must satisfy the wave equation to show that:

$$k_y^2 + k_y'^2 = \omega^2(\varepsilon_1 - \varepsilon_2)\mu_0$$

and then combine that with the result from (b) to show that the dispersion relation for this waveguide is:

$$-\cot k_y d = \sqrt{\frac{\omega^2(\varepsilon_1 - \varepsilon_2)\mu_0}{k_y^2} - 1}$$

d) At this point we have all the relations we need to characterize any particular mode, or to find out how many modes a particular waveguide will support. Let's suppose we have a glass waveguide ($n_2 = 1.5$) with plastic cladding ($n_1 = 1.3$) and a 3 micron radius. Let's also suppose we inject green laser light with a vacuum wavelength of 532 nm. How many modes can exist in our waveguide? Feel free to solve the dispersion relation numerically or graphically, but show your work either way.