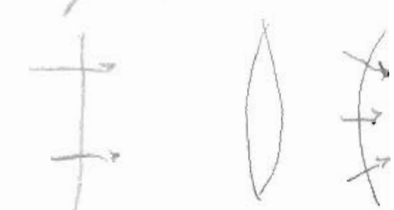
Focusing effects



plane wave
all $\vec{k}$ same

focusing spherical wave.
range of $\vec{k}$'s across beam.

3-D wave equation

$$\nabla^2 \vec{E}_n - \frac{n^2}{c^2}\frac{\partial^2 \vec{E}_n}{\partial t^2} = \frac{4\pi}{c^2}\frac{\partial^2 \vec{P}_n^{NL}}{\partial t^2}$$

remember: LHS is just linear W.E. for each beam.
RHS is source
$\therefore$ all propagation effects matter:
interference, diffraction, focusing ...
In many cases, figure out how the waves propagate linearly, then use these solutions in NL eqn.

Paraxial form: wave is predominantly forward prop.

$$\vec{E}_n(\vec{r},t) = \vec{A}_n(\vec{r})\, e^{i(k_n z - \omega_n t)} + c.c.$$

$$\vec{P}_n^{NL}(\vec{r},t) = \vec{p}_n(\vec{r})\, e^{i(k_n' z - \omega_n t)} + c.c.$$

Impt: k_n is wavevector for plane wave in medium.

Write ∇^2 as $\partial_z^2 + \nabla_T^2$ rect: $\nabla_T^2 = \partial_x^2 + \partial_y^2$

cyl: $\nabla_T^2 = \frac{1}{r}\partial_r(r\partial_r) + \frac{1}{r^2}\partial_\phi^2$

$$\underbrace{-k_n^2 \vec{A}_n + 2ik_n \frac{\partial \vec{A}_n}{\partial z} + \frac{\partial^2 \vec{A}_n}{\partial z^2}} + \nabla_T^2 \vec{A}_n + \frac{n^2\omega_n^2}{c^2}\vec{A}_n = -\frac{4\pi\omega_n^2}{c^2}\vec{p}_n e^{i\Delta k z}$$

compare: $\frac{2\pi}{\lambda_n}\frac{1}{L}A_n : \frac{1}{L^2}A_n$ ∂_z^2 small if
$L \gg \lambda/2\pi$

$$2ik_n \partial_z \vec{A}_n + \underbrace{\left(\frac{n^2\omega_n^2}{c} - k_n^2\right)}_{=0} \vec{A}_n + \nabla_T^2 \vec{A}_n = -4\pi k_{on}^2 \vec{p}_n e^{i\Delta k z}$$

In waveguide, we use $A(x,y)e^{ik_z z}$ as trial solution.

- Here it is $A(x,y,z)e^{ik_{on} z}$.
- Later the function $\phi(z) = \tan^{-1}(z/z_R)$ contains adjustments to this phase.

paraxial wave eqn:

$$2ik_n \partial_z \vec{A}_n + \nabla_T^2 \vec{A}_n = -4\pi \frac{\omega_n^2}{c^2} \vec{p}_n e^{i\Delta k z}$$

Gaussian Beam propagation

one solution to the paraxial W.E. is the Gaussian beam:

$$A(r,z) = \overset{1}{A_0}\ \overset{2}{\frac{w_0}{w(z)}}\ \overset{3}{e^{-r^2/w(z)^2}}\ \overset{4}{e^{-ikr^2/2R(z)}}\ \overset{5}{e^{i\Phi(z)}}$$

with $w(z) = w_0 \left(1 + (z/z_R)^2\right)^{1/2}$ beam radius

$z_R = \pi w_0^2/\lambda$ Rayleigh range.

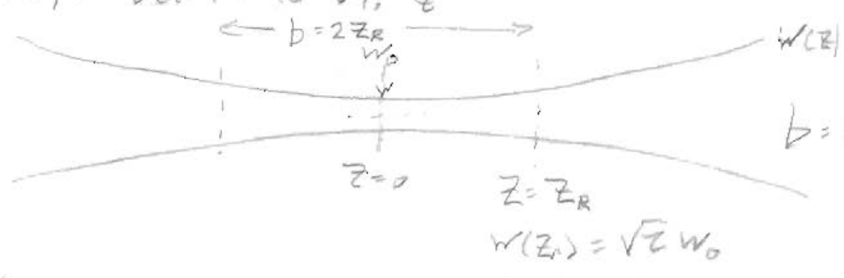
$R(z) = z\left(1 + (z_R/z)^2\right)$ wavefront radius

$\Phi(z) = -\tan^{-1}(z/z_R)$ "Gouy" phase

Pieces:

1) normalization = peak field at focus, $r=0, z=0$

2) $w(z)$: beam rad vs. z



b = confocal param.

$w_0/w(z) \sim E(0,z)$ to account for intensity decr. w/ beam size

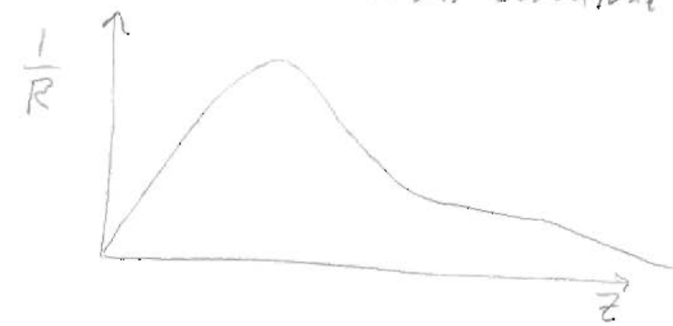
3) gaussian amplitude profile $w \to 1/e$ rad in field.
beam stays gaussian

4) paraxial spherical wave

$$e^{ikR} \to e^{ikz\sqrt{1 - r^2/z^2}} \doteq e^{ikz} e^{-ikr^2/2z^2}$$

let $z \sim R$

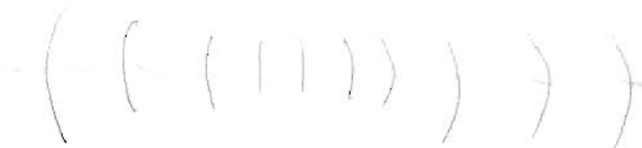
$R(z)$ = radius of wavefront curvature



wavefront is flat in focus and at $z = \pm\infty$

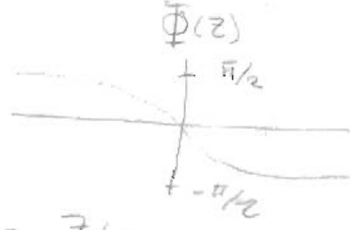
5) Gouy phase: on-axis phase shift

As beam passes through focus, wavefront changes from converging to collim to diverging:



full on-axis phase:

$$e^{i(kz - \tan^{-1}(z/z_R))}$$



for small z/z_0 $\tan^{-1}(z/z_R) \to z/z_R$

$\therefore$ in focus

$$e^{i((k - 1/z_R)z - \omega t)}$$

phase velocity: $\omega/(k - 1/z_R) > \omega/k$ faster.

Complete phase shift through focus: $-\pi$

length scaling:

define $\xi = z/z_R = 2z/b$

see notebook Gaussian beam paraxial.nb for derivation.

$$\to A(r,z) = A_0 \frac{1}{1+i\xi} e^{-r^2/w_0^2(1+i\xi)}$$

Harmonic generation w/ focused beams

- q^{th} harmonic generation $\omega_q = q\omega_1$

put gaussian beam into input A_1

$$2ik_q \frac{\partial A_q}{\partial z} + \nabla_T^2 A_q = -\frac{4\pi \omega_q^2}{c^2} \chi^{(q)} A_1^q e^{i\Delta k z}$$

guess $A_q(r,z)$ has Gaussian form, assume constant pump.

$$A_q(r,z) = \frac{A_{q0}(z)}{1+i\xi_q} e^{-ir^2/w_{q0}^2(1+i\xi_q)}$$

LHS: do derivative, end up with

$$e^{-ir^2/w_{q0}^2(1-i\xi_q)}$$

RHS: $A_1^q \to e^{-iqr^2/w_0^2(1+i\xi)}$

can cancel terms if $w_{q0}^2 = w_0^2/q$ and $z_{Rq} = n_q \frac{\pi w_{q0}^2}{\lambda_q} = n_1 \frac{\pi w_0^2}{q} \frac{q}{\lambda_0}$

same rayleigh range for both beams. $= z_R$

$$\to A_{q0}(z) \propto J_q(\Delta k, z_0, z) = \int_{z_0}^{z} \frac{e^{i\Delta k z'}}{\left(1+i\frac{z'}{z_R}\right)^{q-1}} dz'$$

for odd harmonic generation, $A_{q0}(\infty) = 0$ in tight focusing limit

and $\Delta k = 0$