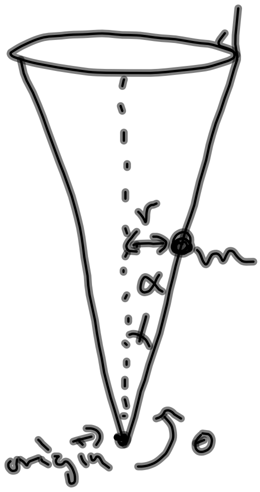
Today: Lagrangian
Friday: Hamiltonian

$\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + \dot{z}\hat{z}$



$\tan\alpha = \frac{r}{z}$

$z = r\cot\alpha$

$\dot{z} = \dot{r}\cot\alpha$

$v^2 = \dot{r}^2 + r^2\dot{\theta}^2 + \dot{r}^2\cot^2\alpha$

$u = mgh = mgr\cot\alpha \}(mgz)$

h goes up as you go up.

$L = T - u = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{r}^2\cot^2\alpha) - mgr\cot\alpha$

$\frac{\partial L}{\partial r} = mr\dot{\theta}^2 - mg\cot\alpha$

$\frac{\partial L}{\partial \dot{r}} = m\dot{r} + m\dot{r}\cot^2\alpha = m\dot{r}(1+\cot^2\alpha)\cdot\frac{\sin^2\alpha}{\sin^2\alpha}$

$= m\dot{r}\csc^2\alpha$

$\underbrace{\frac{\partial L}{\partial \theta} = \emptyset}$; $\frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} = \text{const.} = c_1$

L is cyclic in θ, or θ is a cyclic variable.

because $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = +\frac{\partial L}{\partial \theta}$

Physically this is the angular momentum.

Define generalized momentum $p_\theta = \frac{\partial L}{\partial \theta}$

$p_j = \frac{\partial L}{\partial \dot{q}_j}$

$$\frac{\partial L}{\partial r} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = \varnothing$$

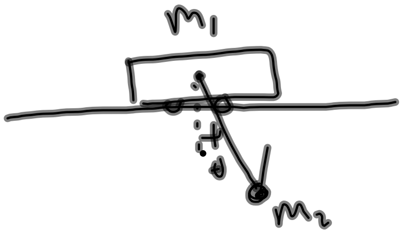
$$m r \dot{\theta}^2 - mg\cot\alpha - m\ddot{r}\csc^2\alpha = \varnothing$$

$$m\ddot{r} - mr\sin^2\alpha\,\dot{\theta}^2 + mg\cos\alpha\sin\alpha = \varnothing$$

$$m\ddot{r} - mr\sin^2\alpha\left(\frac{c_1}{mr^2}\right)^2 + mg\cos\alpha\sin\alpha = \varnothing$$

$$m\ddot{r} - \frac{c_1^2\sin^2\alpha}{mr^3} + mg\cos\alpha\sin\alpha = \varnothing$$

Another:



Coords
$x = x_{m1}$
θ

$\underline{m_1}$

$\vec{v}_1 = \dot{x}\,\hat{x}$

$U_1 = \varnothing$

$\underline{m_2}$

$\vec{x}_{rel} = L\sin\theta\,\hat{x}$

$\vec{y}_{rel} = -L\cos\theta\,\hat{y}$

$\vec{r}_{rel} = L\sin\theta\,\hat{x} - L\cos\theta\,\hat{y}$

$\vec{v}_{rel} = L\cos\theta\,\dot{\theta}\,\hat{x} + L\sin\theta\,\dot{\theta}\,\hat{y}$

$\vec{v}_2 = (\dot{x} + L\cos\theta\,\dot{\theta})\hat{x} + L\sin\theta\,\dot{\theta}\,\hat{y}$

$U_2 = mgh$
$= -m_2 g L\cos\theta$

$\dot{x}^2 + 2\dot{x}L\cos\theta\,\dot{\theta} + L^2\cos^2\theta\,\dot{\theta}^2$

$$T = \tfrac{1}{2}m_1\dot{x}^2 + \tfrac{1}{2}m_2(\dot{x} + L\cos\theta\,\dot{\theta})^2 + \tfrac{1}{2}m_2(L\sin\theta\,\dot{\theta})^2$$
$$= \tfrac{1}{2}(m_1+m_2)\dot{x}^2 + m_2 L\cos\theta\,\dot{x}\dot{\theta} + \tfrac{1}{2}m_2 L^2\dot{\theta}^2$$

$L = T - U$

$$\frac{\partial L}{\partial \theta} = -m_2 g L\sin\theta - m_2 L\sin\theta\,\dot{x}\dot{\theta}$$

$$\frac{\partial L}{\partial \dot{\theta}} = m_2 L\cos\theta\,\dot{x} + m_2 L^2\dot{\theta}$$

$$\frac{d}{dt}(\;) = -m_2 L\sin\theta\,\dot{\theta}\dot{x} + m_2 L\cos\theta\,\ddot{x} + m_2 L^2\ddot{\theta}$$

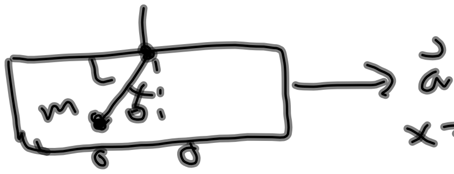
$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = -m_2 g L\sin\theta - m_2 L\cos\theta\,\ddot{x} - m_2 L^2\ddot{\theta} = \varnothing$$

$$\frac{\partial L}{\partial x} = \emptyset \quad \rightarrow \quad \frac{\partial L}{\partial \dot{x}} = \text{const}$$

x is cyclic

$$\frac{\partial L}{\partial \dot{x}} = (m_1+m_2)\dot{x} + m_2 L\cos\theta\,\dot{\theta} = \text{const.}$$

Example:



$$x = x_0 + v_0 t + \tfrac{1}{2} a t^2$$

$$x = x_0 + v_0 t + \tfrac{1}{2} a t^2 - L\sin\theta \qquad U = -mgL\cos\theta$$

$$y = -L\cos\theta$$

$$\dot{x} = v_0 + at - L\cos\theta\,\dot{\theta}$$

$$\dot{y} = L\sin\theta\,\dot{\theta}$$

$$T = \tfrac{1}{2} m\left[(v_0+at - L\cos\theta\,\dot{\theta})^2 + L^2\sin^2\dot{\theta}^2\right]$$

$$L^2\dot{\theta}^2$$

$$L = \tfrac{1}{2} m\left[v_0^2 + a^2t^2 + L^2\cos^2\dot{\theta}^2 - 2(v_0+at)L\cos\theta\,\dot{\theta} + 2v_0 a t + L^2\sin^2\theta\,\dot{\theta}^2\right] + mgL\cos\theta$$

$$\frac{\partial L}{\partial \theta} = m(v_0+\cancel{at})L\sin\theta\,\dot{\theta} - mgL\sin\theta$$

$$\frac{\partial L}{\partial \dot{\theta}} = mL^2\dot{\theta} - m(v_0+at)L\cos\theta$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = mL^2\ddot{\theta} + m(v_0+\cancel{at})L\sin\theta\,\dot{\theta} - mLa\cos\theta$$

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = -mgL\sin\theta - mL^2\ddot{\theta} + mLa\cos\theta$$

$$\ddot{\theta} = \frac{a}{L}\cos\theta - \frac{g}{L}\sin\theta$$