Closed box: blackbody cavity

3-D solution to wave eqn can be written as

$$E \sim A e^{i k_x x} e^{i k_y y} e^{i k_z z} e^{-i\omega t}$$

or

$$A e^{i(\vec{k}\cdot\vec{r} - \omega t)} \qquad \vec{k} \text{ is wave vector}$$

boundary condition for wave along z

$E_x = 0$ at z walls → use sines

$E_x = 0$ along x walls " "

$\partial_y E_x = 0$ along y walls " cos ;

→ $E_x(x,y,z) = A_y \sin k_x x \cos k_y y \sin k_z z$

other permutations work too:

$$A_x \cos k_x x \sin k_y y \sin k_z z$$

$$A_z \sin k_x x \sin k_y y \cos k_z z$$

into wave eqn,

$$\rightarrow \quad k_x^2 + k_y^2 + k_z^2 = \frac{\omega^2}{c^2} = k^2$$

$$\text{with} \quad k_x = \frac{l\pi}{L_x} \quad k_y = \frac{m\pi}{L_y} \quad k_z = \frac{n\pi}{L_z}$$

→ allowed frequencies

$$\omega_{lmn} = c\sqrt{\left(\frac{l\pi}{L_x}\right)^2 + \left(\frac{m\pi}{L_y}\right)^2 + \left(\frac{n\pi}{L_z}\right)^2}$$

$$\text{or} \quad \nu_{lmn} = c\sqrt{\left(\frac{l}{2L_x}\right)^2 + \left(\frac{m}{2L_y}\right)^2 + \left(\frac{n}{2L_z}\right)^2}$$

b/c of permutation, there are [illegible] modes for each [illegible]

Now we suppose that the cavity walls are held at
at a temperature T.

If the EM mode energy is in thermal equilibrium with the walls there is a probability that that mode will be excited: $P(E)$

Original assumption: $P(E) \propto e^{-E/kT}$ Maxwell-Boltzmann

$\rightarrow$ avg. energy of each mode is

$$\langle E \rangle = \frac{\int_0^\infty E\, e^{-E/kT}\, dE}{\int_0^\infty e^{-E/kT}\, dE} = kT$$

this assumes a continuous range of allowed energies.
(will be shown to be wrong)

$\langle E \rangle = kT$ gives energy in each mode.
need # modes/volume of box/unit $d\nu$ = p_ν
- ν is frequency of EM wave (quantized b/c of boundary conditions)

$\therefore$ $p_\nu \langle E \rangle$ = spectral energy density = ρ_ν
so that total energy density is $\int_0^\infty \rho_\nu \, d\nu$

Counting # modes:

in k space, modes are equally spaced along each axis
each combination of $l, m, n \rightarrow \vec{k}$
many $\vec{k}$'s that give same $|\vec{k}|$ or ν
$\therefore$ count # modes in a volume of k-space.

$N(\nu)$ = # modes w/ freq ν or less, $|\vec{k}| < \frac{2\pi\nu}{c}$

$$= 2 \cdot \frac{1}{8} \frac{\frac{4}{3}\pi\left(\frac{2\pi\nu}{c}\right)^3}{\frac{\pi}{2L_x}\frac{\pi}{2L_y}\frac{\pi}{2L_z}}$$

→ vol of sphere in k-space

→ vol. of k-space occup. by 1 mode

↑ polariz

↑ only the $\hat{x}$, $\hat{y}$, $\hat{z}$ segment.

$$= \frac{8\pi\nu^3}{3c^3} V \qquad V = L_x L_y L_z = \text{vol. of box}$$

now $p_\nu = \frac{1}{V}\frac{dN}{d\nu} = \frac{8\pi\nu^2}{c^3}$

As $\nu \uparrow$ many more possible modes.

∴ energy density is

$$\rho_\nu = p_\nu \langle E \rangle = \frac{8\pi\nu^2}{c^3} kT \qquad \text{Rayleigh - Jeans}$$

but total energy density is $\int_0^\infty \frac{8\pi\nu^2}{c^3} kT \, d\nu = \infty$

<u>Planck</u>: energy in each mode is quantized

$E = nh\nu$ not continuous.

∴ $\langle E \rangle$ is a sum not integral

$$= \frac{\sum_n nh\nu \, e^{-nh\nu/kT}}{\sum_n e^{-nh\nu/kT}} = \frac{h\nu}{e^{h\nu/kT} - 1}$$

for $h\nu \to 0$ same as R-J

but as $h\nu > kT$ denominator in $\langle E \rangle$ gets large

∴ black body curve $\rho_\nu = \frac{8\pi\nu^2}{c^3} \frac{h\nu}{e^{h\nu/kT} - 1}$

Comparing w/expt → $h = 6.62 \times 10^{-34}$ Js Planck's constant

B-B curve:

- total amt of radiation ↑ with T (Stefan-Boltzmann)

  $E_{tot} \propto T^4$
- note that curve has diff't peak when measured in λ space.
- BB expr gives photon occupation / mode present in vacuum.

  $$\langle p \rangle \equiv \frac{\langle E \rangle}{h\nu} = \frac{1}{e^{h\nu/kT} - 1}$$

very small at room temp $kT \sim 1/40$ eV for visible light $h\nu \sim 2$eV

Energy density of E&M waves:

$$\rho = \langle \tfrac{1}{2}\varepsilon E^2 \rangle + \langle \tfrac{1}{2}\mu H^2 \rangle$$

↳ time average

return to 1-D resonator:

- allowed λ or ν is quantized by walls (1st quantization) but classically field strength is continuous.

- QED: hamiltonian for field (= energy)

$$\mathcal{H}_\nu = \tfrac{1}{2}\varepsilon E_\nu^2 + \tfrac{1}{2}\mu H_\nu^2$$ for a given cavity mode.

Solve by analogy w/ SHO:

$$\mathcal{H} = \frac{1}{2}\frac{\hat{p}_x^2}{m} + \frac{1}{2}m\omega^2\hat{x}^2 \qquad \hat{p}_x \to E_\nu \qquad \hat{X} \to H_\nu$$

→ discrete allowed energies, separated by $h\nu$

(2nd quantization) $\rho_\nu = (n_\nu + \tfrac{1}{2})h\nu/V$

min energy for $n_\nu = 0$ $= \dfrac{\tfrac{1}{2}h\nu}{V}$ = vac fluctuations.

Laser cavity can put many photons in same mode.

- photons are bosons, follow Bose-Einstein stats