MATH348 - July 2, 2009
Exam I - 50 Points

NAME: Key

In order to receive full credit, SHOW ALL YOUR WORK. Full credit will be given only if all reasoning and work is provided. When applicable, please enclose your final answers in boxes.

1. (10 Points) True/False and Short Response

   (a) Mark each statement True or False. When not explicitly stated take $\mathbf{A} \in \mathbb{R}^{m\times n}$.

      i. If a matrix $\mathbf{A}$ has a row of zeros then $\mathbf{Ax} = \mathbf{0}$ has infinity-many solutions.

      False Consider $\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow$ 1 soln $x_1 = x_2 = 0$

      ii. If $\mathbf{A}$ has a pivot in every row then $\mathbf{Ax} = \mathbf{b}$ has a solution.

      True. Pivot in Every Row implies no Row of $[0\,0\,0 \cdots 0 \mid \# \neq 0]$

      iii. If $\mathbf{A}$ has a pivot in every column then $\mathbf{Ax} = \mathbf{b}$ has a solution.

      False. Consider $\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{array}\right] \Rightarrow$ no Soln

      iv. The system $\mathbf{Ax} = \mathbf{0}$, where $\mathbf{A} \in \mathbb{R}^{3\times 5}$, has only the trivial solution.

      False. 3 Eqns 5 unknown $\Rightarrow$ free vars. $\Rightarrow$ nontrivial Soln

   (b) Please respond to the following questions and justify your position:

      i. Suppose that $\det(\mathbf{A}) = 0$, what can be said about the dimension of the null-space of $\mathbf{A}$ and the dimension of the column-space of $\mathbf{A}$?

      $\det(A) = 0 \Rightarrow$ free variables $\Rightarrow \dim(\text{Nul}(A)) \neq 0$

      $\Rightarrow \dim(\text{Col}(A)) <$ # of Columns

      ii. Is it possible for a vector to be in both the null-space and the column-space of a matrix?

      Yes. $\vec{0}$ is always in both spaces.
      See HW#3 problem 3 for nontrivial Example.

2. (10 Points) Quickies:

(a) Given,

$$\mathbf{A}_1 = \begin{bmatrix} 8 & 0 & 8 \\ 0 & 8 & 0 \\ 8 & 0 & 8 \end{bmatrix} \quad \mathbf{A}_2 = \begin{bmatrix} 0 & 8 & 0 \\ 8 & 0 & 8 \\ 0 & 8 & 0 \end{bmatrix} \quad \mathbf{b}_1 = \begin{bmatrix} 8 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{b}_2 = \begin{bmatrix} 0 \\ 8 \\ 0 \end{bmatrix} \quad \mathbf{b}_3 = \begin{bmatrix} 0 \\ 0 \\ 8 \end{bmatrix}$$

Do solutions to the following equations exist and are these solutions unique? (Yes or No)

Equation	Solutions Exist	Solutions are Unique
$\mathbf{A}_1\mathbf{x} = \mathbf{b}_1$	No	No
$\mathbf{A}_2\mathbf{x} = \mathbf{b}_2$	Yes	No
$\mathbf{A}_1\mathbf{x} = \mathbf{b}_1 + \mathbf{b}_3$	Yes	no
$\mathbf{A}_2\mathbf{x} = \mathbf{b}_1 - \mathbf{b}_3$	~~Yes~~ No	no
$(\mathbf{A}_1 + \mathbf{A}_2)\,\mathbf{x} = \mathbf{b}_1 + \mathbf{b}_2 + \mathbf{b}_3$	Yes	no

(b) Let $\mathbf{A} = \begin{bmatrix} 1 & h \\ 3 & 4 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 3 \\ h \end{bmatrix}$. Find **all** values h for which the system $\mathbf{A}\mathbf{x} = \mathbf{b}$ is:

$[A \mid \vec{b}] \sim \left[\begin{array}{cc|c} 1 & h & 3 \\ 0 & 4-3h & h-9 \end{array}\right]$

i. Consistent with a unique solution.

$4 - 3h \neq 0 \Rightarrow h = \frac{4}{3}$

ii. Consistent with infinity-many solutions.

Impossible

iii. Inconsistent.

$h = \frac{4}{3}$

(c) Given,

$$\mathbf{A} = \begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}. \tag{1}$$

Find one eigenvalue of $\mathbf{A}$.

$$\begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \Rightarrow A\vec{x} = \lambda\vec{x} \text{ where } \boxed{\lambda = 1}$$

3. (10 Points) Given,

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -3 \\ 9 \\ -6 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 5 \\ -7 \\ h \end{bmatrix}.$$

For what values of h are the vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ a linearly dependent?

$$\begin{bmatrix} 1 & -3 & 5 \\ -3 & 9 & -7 \\ 2 & -6 & h \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 5 \\ 0 & 0 & 8 \\ 0 & 0 & h-10 \end{bmatrix}$$

$\vec{V}_3$ is lin. ind. of $\vec{v}_1, \vec{V}_2$ for any h.
However $\vec{V}_2$ is always lin. dep to $\vec{V}_1$
$\Rightarrow \{\vec{v}_1, \vec{V}_2, \vec{V}_3\}$ is lin dep.
$\forall h \in \mathbb{R}$

4. (10 Points) Given,

$$\mathbf{A} = \begin{bmatrix} 4 & 0 & -2 \\ 2 & 5 & 4 \\ 0 & 0 & 5 \end{bmatrix}.$$

Find all matrices associated with the diagonal decomposition, $\mathbf{A} = \mathbf{PDP}^{-1}$, of $\mathbf{A}$.

$$\det(A - \lambda I) = \left| \begin{bmatrix} 4-\lambda & 0 & -2 \\ 2 & 5-\lambda & 4 \\ 0 & 0 & 5-\lambda \end{bmatrix} \right| = (4-\lambda)(5-\lambda)^2 - 2\cdot 0 = 0$$

$$\Rightarrow \lambda_1 = 4, \; \lambda_2 = 5 = \lambda_3$$

Case $\lambda_1 = 4$:

$$\left[\begin{array}{ccc|c} 0 & 0 & -2 & 0 \\ 2 & 1 & 4 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \Rightarrow 2x_1 = -x_2$$

$$\Rightarrow \vec{x}^{(1)} = \begin{bmatrix} x_1 \\ -2x_1 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix},$$

Case $\lambda_2 = \lambda_3 = 5$:

$$\left[\begin{array}{ccc|c} -1 & 0 & 2 & 0 \\ 2 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \vec{x} = \begin{bmatrix} 2x_3 \\ x_2 \\ x_3 \end{bmatrix}$$

$\Rightarrow x_1 = 2x_3, \; x_3 \in \mathbb{R}$
$x_2 \in \mathbb{R}$

$$\vec{x}^{(2)} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{x}^{(3)} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 0 & 2 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ -2 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 4 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -2 \\ 0 & 1 & 0 & 2 & 1 & -4 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \Rightarrow P^{-1} = \begin{bmatrix} 1 & 0 & -2 \\ 2 & & \\ 0 & 0 & 1 \end{bmatrix} \leftarrow 1 \; -4$$

5. (10 Points) Given,

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix} \qquad (2)$$

Find a basis and the dimension of the null-space and column-space of $\mathbf{A}$.

$x_2 = -x_4$

$x_3 = 0$

$x_1 = -x_5$

$x_2 + x_3 + x_4 = -x_4 + 0 + x_4 = 0$

$B_{Col(A)} = \left\{ \begin{bmatrix} 0\\0\\0\\1\\0 \end{bmatrix}, \begin{bmatrix} 0\\1\\0\\0\\1 \end{bmatrix}, \begin{bmatrix} 0\\0\\1\\0\\1 \end{bmatrix} \right\}$

$\dim\{Col(A)\} = 3$

$\vec{x} = \begin{bmatrix} -x_5\\-x_4\\0\\x_4\\x_5 \end{bmatrix} = \begin{bmatrix} -1\\0\\0\\0\\1 \end{bmatrix} x_5 + \begin{bmatrix} 0\\-1\\0\\1\\0 \end{bmatrix} x_4 \Rightarrow B_{\cancel{Row(A)}\,Nul} = \left\{ \begin{bmatrix} -1\\0\\0\\0\\1 \end{bmatrix}, \begin{bmatrix} 0\\-1\\0\\1\\0 \end{bmatrix} \right\}$ $\dim(Nul(A)) = 2$

6. (Extra Credit 1) Given,

$$[\mathbf{A}|\mathbf{0}] = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]. \qquad (3)$$

List as many properties/characterizations of this system as you can for one point each and six points maximum.

1. $A\vec{x} = \vec{0}$ has nontrivial Sol$\underline{n}$
2. col. 3 is lin.dep. to col. 1, 2.
3. $\dim\{Col(A)\} = 2$
4. $\dim(Nul(A)) = 1$
5. $A\vec{x} = \vec{b}$ is incon. for some $\vec{b} \in \mathbb{R}^5$
6. Columns do not span $\mathbb{R}^5$

7. (Extra Credit 2) Concerning the matrix in problem 5 of this exam.

(a) Is the matrix invertible? Justify your choice.

No. Row of zeros $\Rightarrow$ $\det(A) = 0 \Rightarrow A^{-1}$ does not exist.

(b) Can only one element of the matrix be changed so that its invertibility is altered?

No. 2 more pivots are needed. Only 1 can be introduced.

(c) Is this matrix happy?

Yes! :)

(d) Can you change the mood of this matrix with one elementary row-step?

~~R~~ RowSwap: Row 4 $\leftrightarrow$ Row 5 $\Rightarrow$