MATH 225 - Differential Equations May 21, 2008
Homework 4, Field 2008 **Not Due Date**: May 22, 2008

BIFURCATIONS - LINEARITY - UNDETERMINED COEFFICIENTS - INTEGRATING FACTORS

1. For the one-parameter family $\dfrac{dy}{dt} = y^2 - ay + 4, \quad y, a \in \mathbb{R}$,
   (a) Find the bifurcation value(s).
   (b) For each bifurcation value, draw phase lines for values of the parameter slightly smaller than, slightly larger than, and at the bifurcation value. Make sure to label your graph and classify any equilibrium points.

2. Given that $\dfrac{dy}{dt} = a(t)y$. Show that if $y_1(t)$ and $y_2(t)$ are both solutions to the homogenous linear differential equation then the linear combination $y(t) = c_1y_1(t) + c_2y_2(t)$ where $c_1, c_2 \in \mathbb{R}$ is also a solution.[1]

3. Determine the general solution for the following linear differential equations:

   (a) $\dfrac{dy}{dt} - 2y = t^2 + 3e^t$ (b) $y' = 5y + 3e^{5t}$

   (c) $y' = -3y + 2\cos(2t)$ (d) $y'' - 3y' + 2y = 0$

   Hint: For (d) assume that $y(t) = e^{rt}$, $r \in \mathbb{R}$, and show that $y'' - 3y' + 2y = 0 \iff r^2 - 3r + 2 = 0$. Solve for r to find two possible solutions. In this case the general solution is $y(t) = c_1y_1(t) + c_2y_2(t), \ \ c_1, c_2 \in \mathbb{R}$.

4. Using integrating factors, solve the following differential equation or initial-value problem.
   (a) $\dfrac{dy}{dt} = \dfrac{t^3y}{1+t^4} + 2t^3$
   (b) $\dfrac{1}{2t}\dfrac{dy}{dt} = y + \dfrac{3}{2}e^{t^2}, \quad y(0) = 1$

5. A 1000 gallon tank initially contains a mixture of 450 gallons of cola and 50 gallons of cherry syrup. Cola is added at the rate of 8 gallons per minute, and cherry syrup is added at the rate of 2 gallons per minute. At the same time, a well mixed solution of cherry cola is withdrawn at the rate of 5 gallons per minute.
   (a) Write down the initial-value problem that models the rate of cherry syrup flowing through the tank.
   (b) Solve the initial-value problem.
   (c) When is the tank full?
   (d) How much cherry syrup is in the tank when it is full?

[1] This result is true for any homogenous linear differential equation.

MATH225, Spring 2008 **Name:** Solutions
Worksheet 4 (1.6, 1.7, 1.8)

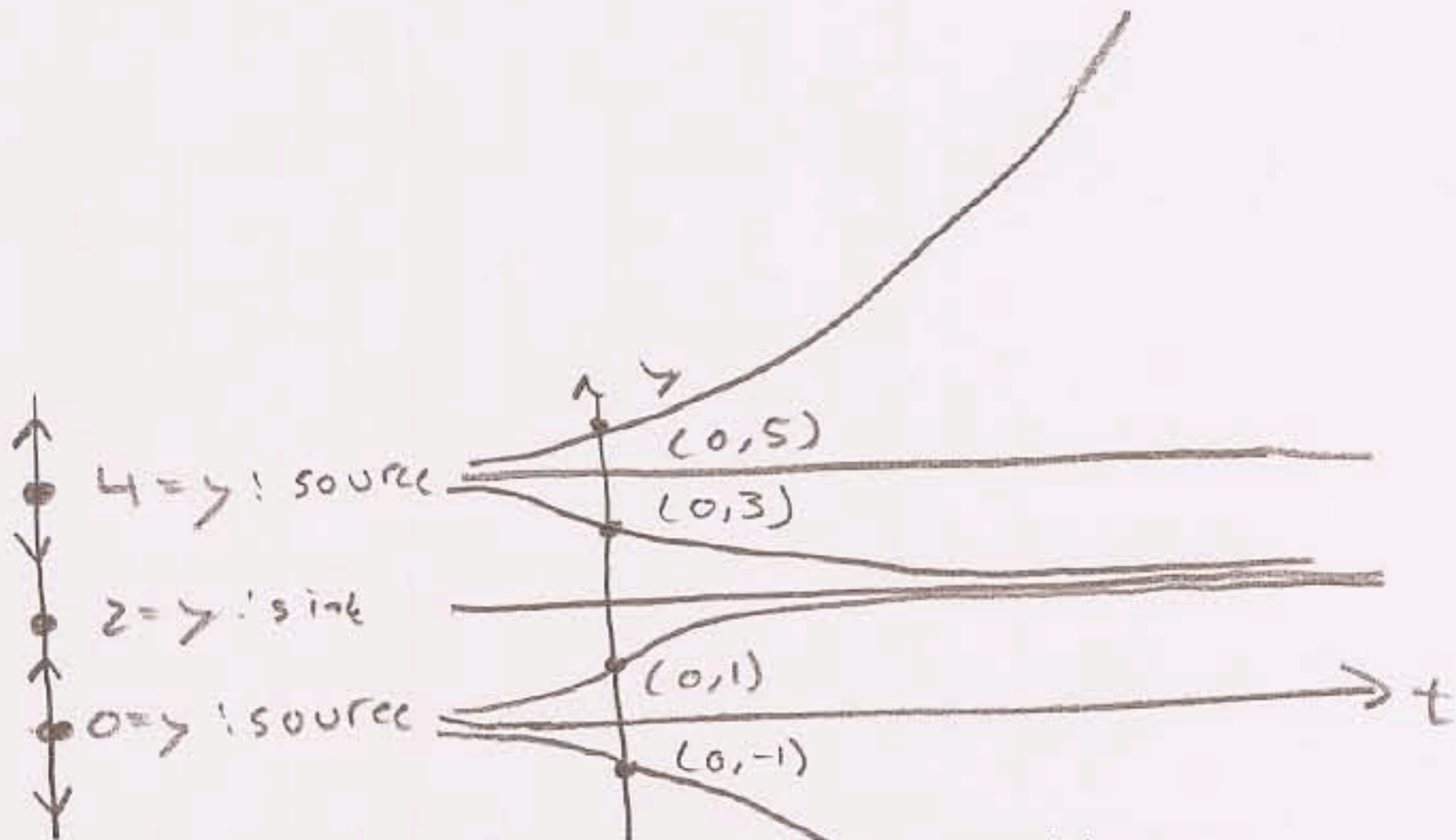
For full credit, you must show all work and box answers.

1. Given $\frac{dy}{dt} = y(y-2)(y-4)$,

   (a) Sketch the phase line and classify all equilibrium points.

   EP: $y(y-2)(y-4)=0$

   $\boxed{y=0,\ y=2,\ y=4}$



   (b) Next to your phase line, sketch the graphs of solutions satisfying the initial conditions $y(0) = -1$, $y(0) = 1$, $y(0) = 3$, and $y(0) = 5$. Put your graphs on one pair of axes.

   (c) Describe the long-term behavior of the solution that satisfies the initial condition $y(0) = 1$.

   $\boxed{y \to 2 \text{ as } t \to \infty,\ y \to 0 \text{ as } t \to -\infty}$

2. Given $\frac{dy}{dt} = \sin(y^2)$,

   (a) Sketch the phase line and classify all equilibrium points. (You cannot sketch the entire phase line in this case, but show at least 5 equilibrium points, including both positive and negative y-values.)

   EP: $\sin(y^2)=0$

   $y^2 = 0, \pi, 2\pi, \dots$

   $\boxed{\begin{array}{l} y = 0, \pm\sqrt{\pi}, \pm\sqrt{2\pi}, \dots \\ y = \pm\sqrt{k\pi},\ k=0,1,2,\dots \end{array}}$

   Linearization Thm

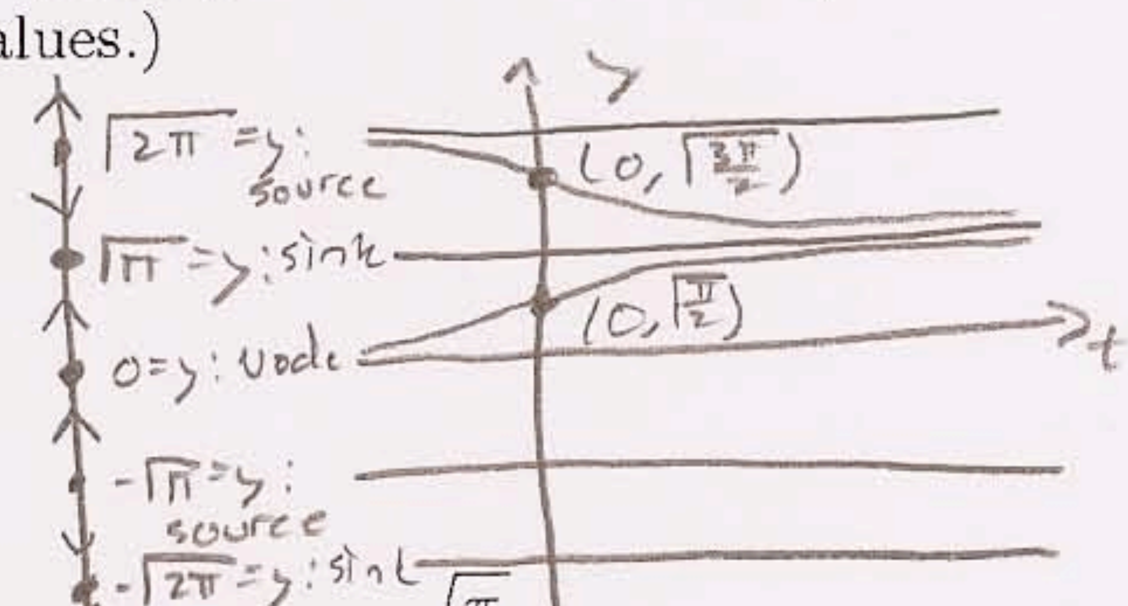
   $f(y) = \sin(y^2)$

   $f'(y) = 2y\cos(y^2)$

   $f'(0) = 0$: ?

   $f'(\sqrt{\pi}) < 0$: sink, $f'(-\sqrt{\pi}) > 0$: source

   $f'(\sqrt{2\pi}) > 0$: source, $f'(-\sqrt{2\pi}) < 0$: sink



   (b) Next to your phase line, sketch the graphs of solutions satisfying the initial conditions $y(0) = \sqrt{\frac{\pi}{2}}$, and $y(0) = \sqrt{\frac{3\pi}{2}}$. Put your graphs on one pair of axes.

   (c) Describe the long-term behavior of the solution that satisfies the initial condition $y(0) = \sqrt{\frac{\pi}{2}}$.

   $\boxed{y \to \sqrt{\pi} \text{ as } t \to \infty,\ y \to 0 \text{ as } t \to -\infty}$

3. For the one-parameter family $\frac{dy}{dt} = y^2 - ay + 4$, $y \in \mathbb{R}$

   (a) Find the bifurcation value(s).

   $f_a(y) = y^2 - ay + 4$

   $f_a'(y) = 2y - a = 0$

   $y = \frac{a}{2}$

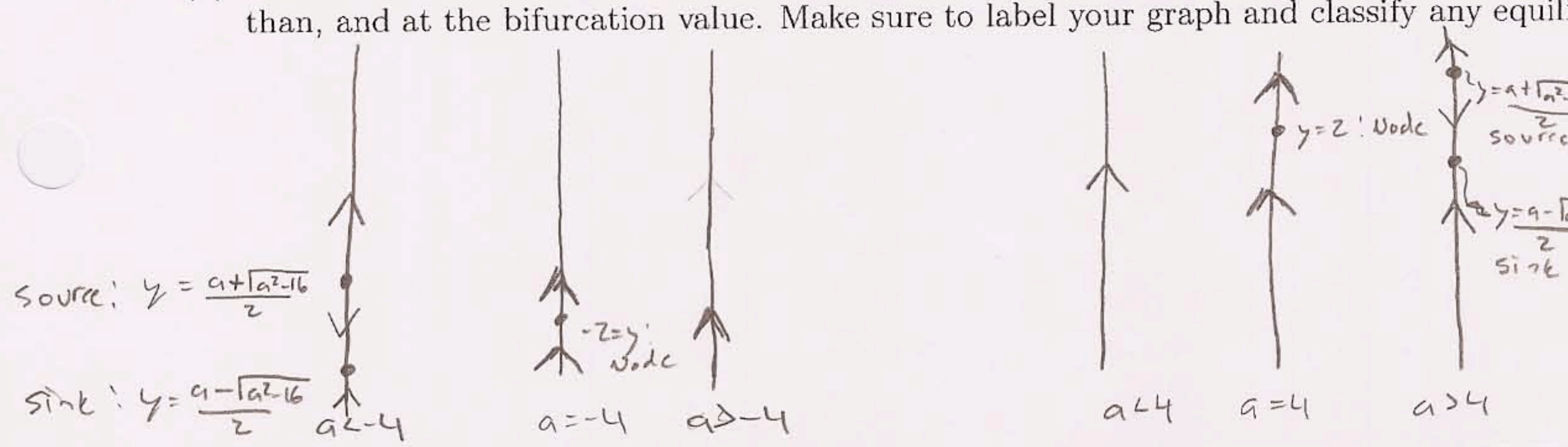
   $f_a(\frac{a}{2}) = \frac{a^2}{4} - \frac{a^2}{2} + 4 = 0$, $\frac{a^2}{4} = 4$, $a^2 = 16$, $\boxed{a = \pm 4}$

   EP: $\frac{dy}{dt} = y^2 - ay + 4 = 0$

   $y = \frac{a \pm \sqrt{a^2-16}}{2}$

   (See attached for classification work.)

   (b) For each bifurcation value, draw phase lines for values of the parameter slightly smaller than, slightly larger than, and at the bifurcation value. Make sure to label your graph and classify any equilibrium points.



3. (cont.)

<u>EP</u>: $f_a(y) = y^2 - ay + 4 = 0$

$$y = \frac{a \pm \sqrt{a^2 - 16}}{2}$$

$a = 4$: $y = \frac{4}{2} = 2$: 1 EP

$$f_a(y) = y^2 - 2y + 4 = (y-2)^2 > 0$$

$-4 < a < 4$: $a^2 - \cancel{16 < 0}$: No EP

$$f_a(y) = y^2 - ay + 4$$

$a = 3,\ y = 1$: $f_3(1) = 1 - 3 + 4 = 2 > 0$

* will not change for any y, since no EP, or for any a until you hit next bifuration value.

$a < -4,\ a > 4$: $y = \frac{a \pm \sqrt{a^2-16}}{2}$: 2 EP

$$f'_a(y) = 2y - a$$

$f'_a\left(\frac{a + \sqrt{a^2-16}}{2}\right) = +\sqrt{a^2-16} > 0$: source

$f'_a\left(\frac{a - \sqrt{a^2-16}}{2}\right) = -\sqrt{a^2-16} < 0$: sink

$a = -4$: $y = \frac{-4}{2} = -2$: 1 EP

$$f_a(y) = y^2 + 2y + 4 = (y+2)^2 > 0$$

3. a. $\frac{dy}{dt} = 2y + t^2 + 3e^t \Rightarrow y' = 2y$ is homogeneous problem

$y_h(t) = e^{2t}$

For nonhomogeneous choose

$y_p(t) = At^2 + Bt + C + De^t$

$y_p'(t) = 2At + B + De^t$

$\Rightarrow$

$$\frac{dy}{dt} = 2At + B + De^t = 2At^2 + 2Bt + 2C + 2De^t + t^2 + 3e^t$$

e^t: $D = 2D + 3 \Rightarrow D = -3$

t^2: $2A + 1 = 0 \Rightarrow A = -1/2$

t: $2A - 2B = 0 \Rightarrow B = +A = -1/2$

const: $B = 2C \Rightarrow C = -1/4$

$$\Rightarrow y(t) = ke^{2t} + -\frac{1}{2}t^2 - 1/2\,t - 1/4 - 3e^t, \quad k \in \mathbb{R}$$

b. $y' = 5y + 3e^{5t}, \Rightarrow y' = 5y \Rightarrow y_h(t) = e^{5t}$

Choose $y_p(t) = Ate^{5t}$

$y_p'(t) = Ae^{5t} + 5Ate^{5t}$

$\Rightarrow$

$$Ae^{5t} + \cancel{5Ate^{5t}} = \cancel{5Ate^{5t}} + 3e^{5t}$$

$A = 3$

$$\Rightarrow y(t) = ke^{5t} + 3te^{5t}, \quad k \in \mathbb{R}$$

c. $y' = -3y + 2\cos(2t) \Rightarrow y' = -3y \Rightarrow y_h(t) = e^{-3t}$

Choose $y_p(t) = A\cos(2t) + B\sin(2t)$

$y_p'(t) = -2A\sin(2t) + 2B\cos(2t)$

$\Rightarrow$

$-2A\sin(2t) + 2B\cos(2t) = -3A\cos(2t) - 3B\sin(2t) + 2\cos(2t)$

$\Rightarrow -2A = -3B \quad , \quad 2B = -3A + 2$

$A = \frac{3}{2}B \Rightarrow 2B = -3\left(\frac{3}{2}A\right) + 2 = -\frac{9}{2}A + 2$

$\frac{4}{2}B + \frac{9}{2}B = 2 \Leftrightarrow \frac{13}{2}B = 2 \Rightarrow B = \frac{4}{13}$

$\Rightarrow A = \frac{6}{13}$

$\Rightarrow Y(t) = ke^{-3t} + \frac{6}{13}\cos(2t) + \frac{4}{13}\sin(2t).$

d. Assume $y(t) = e^{rt}$ then

$y'' - 3y' + 2y = r^2e^{rt} - 3re^{rt} + 2e^{rt} = e^{rt}(r^2 - 3r + 2) = 0$

$\Rightarrow r^2 - 3r + 2 = (r-2)(r-1) = 0 \Rightarrow r = 2, r = 1$

$\Rightarrow y_1(t) = e^{2t}, y_2(t) = e^{t} \Rightarrow y(t) = c_1e^{2t} + c_2e^{t}$

MATH225, Spring 2008 **Name:** Solutions
Worksheet 5 (1.9, 2.1)

For full credit, you must show all work and box answers.

1. Solve the following differential equation or initial-value problem.

(a) $\frac{dy}{dt} = \frac{t^3 y}{1+t^4} + 2t^3$

$\frac{dy}{dt} + \left(\frac{-t^3}{1+t^4}\right)y = 2t^3$

u-sub
$u = 1+t^4$
$du = 4t^3 dt$
$\frac{1}{4}du = t^3 dt$

$\mu(t) = e^{\int \frac{-t^3}{1+t^4}dt}$
$= e^{-\frac{1}{4}\ln(1+t^4)}$
$= e^{\ln[(1+t^4)^{-\frac{1}{4}}]}$
$= (1+t^4)^{-\frac{1}{4}}$
$= \frac{1}{(1+t^4)^{\frac{1}{4}}}$

$\frac{1}{(1+t^4)^{\frac{1}{4}}}\frac{dy}{dt} + \left(\frac{-t^3}{(1+t^4)^{\frac{5}{4}}}\right)y = \frac{2t^3}{(1+t^4)^{\frac{1}{4}}}$

$\frac{d}{dt}\left(\frac{1}{(1+t^4)^{\frac{1}{4}}}y\right) = \frac{2t^3}{(1+t^4)^{\frac{1}{4}}}$

$\frac{1}{(1+t^4)^{\frac{1}{4}}}y = \int \frac{2t^3}{(1+t^4)^{\frac{1}{4}}}dt$

$\frac{1}{(1+t^4)^{\frac{1}{4}}}y = 2\left(\frac{1}{4}\right)\left(\frac{4}{3}\right)(1+t^4)^{\frac{3}{4}} + k$

$\boxed{y(t) = \frac{2}{3}(1+t^4) + k(1+t^4)^{\frac{1}{4}}}$

(b) $\frac{1}{2t}\frac{dy}{dt} = y + \frac{3}{2}e^{t^2}, \quad y(0) = 1$

$\frac{dy}{dt} + (-2t)y = 3te^{t^2}$

$\mu(t) = e^{\int(-2t)dt}$
$= e^{-t^2}$

$e^{-t^2}\frac{dy}{dt} + (-2t)e^{-t^2}y = 3t$

$\frac{d}{dt}(e^{-t^2}y) = 3t$

$e^{-t^2}y = \int 3t\,dt$

$e^{-t^2}y = \frac{3t^2}{2} + k$

$y(t) = e^{t^2}\left(\frac{3t^2}{2} + k\right)$

$y(0) = e^0(0+k) = 1$
$k = 1$

$\boxed{y(t) = e^{t^2}\left(\frac{3t^2}{2} + 1\right)}$

2. Consider the following predator-prey model:

$$\frac{dx}{dt} = \frac{1}{2}x\left(1 - \frac{x}{2}\right) - xy$$
$$\frac{dy}{dt} = \frac{1}{3}y\left(1 - \frac{y}{3}\right) + \frac{1}{12}xy$$

(a) Which variable represents the prey and which represents the predator? Why?

x: prey, interaction term negative $(-xy)$

y: predator, interaction term positive $(+\frac{1}{12}xy)$

(b) What are the equilibrium points of the system?

$\frac{dx}{dt} = x\left[\frac{1}{2}\left(1-\frac{x}{2}\right) - y\right] = 0$

$x = 0$ $\quad \frac{1}{2}\left(1-\frac{x}{2}\right) - y = 0$
$y = \frac{1}{2}\left(1-\frac{x}{2}\right)$
$y = \frac{1}{2} - \frac{x}{4}$

$\frac{dy}{dt} = y\left[\frac{1}{3}\left(1-\frac{y}{3}\right) + \frac{x}{12}\right] = 0$

$x=0$: $y\left[\frac{1}{3}\left(1-\frac{y}{3}\right)\right] = 0$
$y = 0 \quad \frac{1}{3}\left(1-\frac{y}{3}\right) = 0$
$y = 3$

$y = \frac{1}{2} - \frac{x}{4}$: $\left(\frac{1}{2} - \frac{x}{4}\right)\left[\frac{1}{3}\left(1 - \frac{1}{2} + \frac{x}{12}\right) + \frac{x}{12}\right] = 0$

$\left(\frac{1}{2} - \frac{x}{4}\right)\left[\frac{5}{18} + \frac{x}{9}\right] = 0$

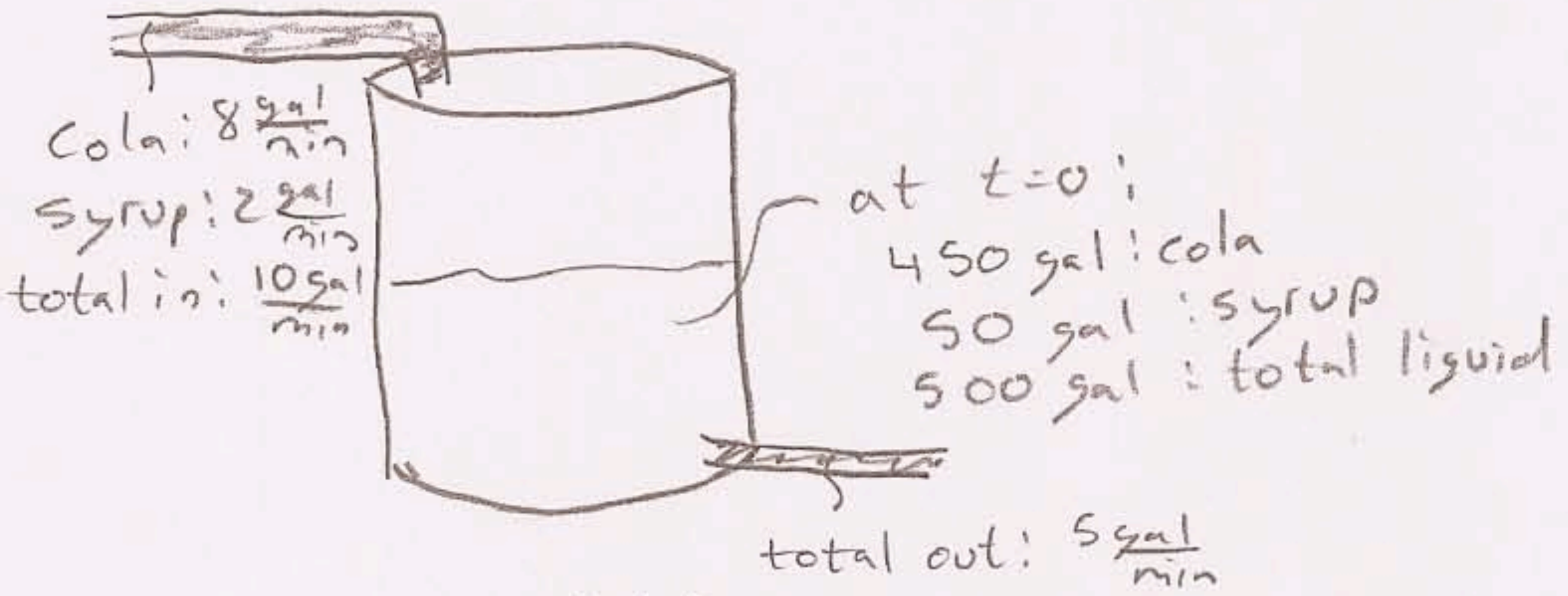
$\frac{1}{2} - \frac{x}{4} = 0$
$x = 2 \Rightarrow y = \frac{1}{2} - \frac{1}{2} = 0$

$\frac{5}{18} + \frac{x}{9} = 0$
$x = -\frac{5}{2} \Rightarrow y = \frac{1}{2} + \frac{5}{8} = \frac{9}{8}$

$\boxed{\text{EP: } (0,0), (0,3), (2,0), \left(-\frac{5}{2}, \frac{9}{8}\right)}$

3. A 1000 gallon tank initially contains a mixture of 450 gallons of cola and 50 gallons of cherry syrup. Cola is added at the rate of 8 gallons per minute, and cherry syrup is added at the rate of 2 gallons per minute. At the same time, a well mixed solution of cherry cola is withdrawn at the rate of 5 gallons per minute.

(a) Write down the initial-value problem that models the dynamics of the tank.



Amt of liquid in tank at time t:

$500 + 5t$ (gal.)

t: time (min)

y: amount of syrup (gal)

$\frac{dy}{dt}$: Rate of syrup flowing through tank ($\frac{gal}{min.}$)

$$\frac{dy}{dt} = 2 - \frac{y}{500+5t}(5)$$

$$\boxed{\frac{dy}{dt} = 2 - \frac{y}{100+t}, \quad y(0) = 50}$$

(b) Solve the initial-value problem.

$$\frac{dy}{dt} + \left(\frac{1}{100+t}\right)y = 2$$

$$\mu(t) = e^{\int \frac{1}{100+t}\,dt}$$

$$= e^{\ln(100+t)}$$

$$= 100+t$$

$$(100+t)\frac{dy}{dt} + y = 2(100+t)$$

$$\frac{d}{dt}\left((100+t)y\right) = 2(100+t)$$

$$(100+t)y = \int 2(100+t)\,dt$$

$$y = \frac{(100+t)^2 + k}{100+t}$$

$$y = 100+t + \frac{k}{100+t}$$

$$y(0) = 100 + \frac{k}{100} = 50, \quad k = -5000$$

$$\boxed{y(t) = 100 + t - \frac{5000}{100+t}}$$

(c) When is the tank full?

$$500 + 5t = 1000$$

$$5t = 500$$

$$\boxed{t = 100 \text{ min.}}$$

(d) How much cherry syrup is in the tank when it is full?

$$y(100) = 200 - \frac{5000}{200} = 200 - 25$$

$$= \boxed{175 \text{ gal.}}$$

4. Consider the equation $\frac{d^2y}{dt^2} + \frac{k}{m}y = 0$ for the motion of a simple harmonic oscillator. Under what conditions on β is $y(t) = \cos(\beta t)$ a solution?

$$y(t) = \cos(\beta t)$$

$$\frac{dy}{dt} = -\beta\sin(\beta t)$$

$$\frac{d^2y}{dt^2} = -\beta^2\cos(\beta t)$$

$$\frac{d^2y}{dt^2} + \frac{k}{m}y = 0$$

$$-\beta^2\cos(\beta t) + \frac{k}{m}\cos(\beta t) = 0$$

$$\left(-\beta^2 + \frac{k}{m}\right)\cos(\beta t) = 0$$

$$-\beta^2 + \frac{k}{m} = 0, \quad \boxed{\beta = \pm\sqrt{\frac{k}{m}}}$$