The Fourier Transform

Heuristic Derivation from Fourier Series

March 26, 2010

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Overview/Keywords/References

Advanced Engineering Mathematics **Slide Set Four**

Nonperiodic Integrable Functions, Fourier Integral, Fourier Transform

Examples:N/A

- See Also:
  - EK : 11.7 - 11.9
  - 11.LN.FourierIntegralandTransforms.pdf
  - HW.6.1
- Begin:
  - Homework 6

Before We Begin

Quote of Slide Set Five
These two are the same but diverge in name as they issue forth. Being the same they are called mysteries, mystery upon mystery - the gateway of the manifold of secrets.
Tao Te Ching : Laozi (late 4^{th} or early 3^{rd} centuries BC)

Abstract

A Fourier series is a very general mathematical construct, which provides an expansion for **any** reasonable periodic function or function that can be represented by a periodic extension. While this class of functions is quite large, it would be nice if these methods could be applied to functions, which are not nor can be made periodic. Heuristically, we consider,

$$f_L(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right), \qquad (1)$$

in the limit $L \to \infty$. Suspension of disbelief and a dint of algebra reveals the following Fourier transform pair,

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{-i\omega x} d\omega \iff \hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx.$$

Definitions and Assumptions

The following quantities will make our derivation cleaner:

- Angular Frequency : $\omega_n = \frac{n\pi}{L}$
- Angular Frequency Step Size : $\triangle\omega = \omega_{n+1} - \omega_n = \frac{\pi}{L}$
- Length Parameter : $\frac{1}{L} = \frac{\triangle\omega}{\pi}$

The following assumption will be required for the derivation:

- Absolute Integrability : $\int_{-\infty}^{\infty} |f(x)|dx < \infty$
  - A consequence of this is that $\lim_{x\to\pm\infty} f(x) = 0$

Full Representation

First we begin with the whole Fourier series:

$$f_L(x) = \frac{1}{2L}\int_{-L}^{L} f(v)dv + \sum_{n=1}^{\infty}\left[\frac{1}{L}\int_{-L}^{L} f(v)\cos(\omega_n v)dv\right]\cos(\omega_n x) + \tag{2}$$

$$+ \left[\frac{1}{L}\int_{-L}^{L} f(v)\sin(\omega_n v)dv\right]\sin(\omega_n x) \tag{3}$$

$$= \frac{1}{2L}\int_{-L}^{L} f(v)dv + \sum_{n=1}^{\infty}\left[\frac{1}{\pi}\int_{-L}^{L} f(v)\cos(\omega_n v)dv\right]\cos(\omega_n x)\triangle\omega \tag{4}$$

$$+ \left[\frac{1}{\pi}\int_{-L}^{L} f(v)\sin(\omega_n v)dv\right]\sin(\omega_n x)\triangle\omega \tag{5}$$

Limit to Fourier Integral

We now consider the limit where $L \to \infty$. Under this limit:

- Step Size : $\triangle\omega \to 0$
- Riemann Sum : $\sum_{n=1}^{\infty} \cos(\omega_n x)\triangle\omega = \int_0^{\infty} \cos(\omega x) d\omega$

In this limit we have the so-called Fourier integral representation of a non-periodic function,

$$f(x) = \lim_{L\to\infty} f_L(x) = \int_0^{\infty} \left[A(\omega)\cos(\omega x) + B(\omega)\sin(\omega x)\right] d\omega, \quad (6)$$

and coefficients,

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos(\omega x)dx, \quad B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin(\omega x)dx \quad (7)$$

Fourier Transform - Part I

From the Fourier integral it is possible to derive the well-celebrated Fourier transform. Consider,

$$f(x) = \int_0^\infty \left[\frac{1}{\pi}\int_{-\infty}^\infty f(v)\cos(\omega v)dv\right]\cos(\omega x)d\omega +$$

$$+ \int_0^\infty \left[\frac{1}{\pi}\int_{-\infty}^\infty f(v)\sin(\omega v)dv\right]\sin(\omega x)d\omega$$

$$= \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(x)\left[\cos(\omega v)\cos(\omega x) + \sin(\omega v)\sin(\omega x)\right]dvd\omega$$

$$= \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(x)\cos(\omega v - \omega x)dvd\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^\infty \int_{-\infty}^\infty f(x)\cos(\omega v - \omega x)dvd\omega -$$

$$- \frac{i}{2\pi}\int_{-\infty}^\infty \int_{-\infty}^\infty f(x)\sin(\omega v - \omega x)dvd\omega$$

Fourier Transform - Part II

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x) \left[\cos(\omega v - \omega x) - i \sin(\omega v - \omega x) \right] dv d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x) e^{-i(\omega v - \omega x)} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(v) e^{-i\omega v} dv \right] e^{i\omega x} d\omega,$$

which defines the following Fourier transform pair,

$$\hat{f}(\omega) = \mathfrak{F}\{f\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx \tag{8}$$

$$f(x) = \mathfrak{F}^{-1}\left\{\hat{f}\right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x} d\omega. \tag{9}$$

Connection to Fourier Series

The Fourier transform and Fourier series are similar in form,

$$f(x) = \frac{1}{\sqrt{2L}} \sum_{n=-\infty}^{\infty} c(\omega_n) e^{i\omega_n x} \iff c(\omega_n) = \frac{1}{\sqrt{2L}} \int_{-L}^{L} f(x) e^{i\omega_n x} dx,$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x} d\omega \iff \hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx,$$

but the Fourier transform is more general. Consider,

$$\mathfrak{F}^{-1}\left\{ \sum_{n=-\infty}^{\infty} \sqrt{2\pi} c_n \delta(\omega - n) \right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \sqrt{2\pi} c_n \delta(\omega - n) e^{i\omega x} d\omega$$

$$= \sum_{n=-\infty}^{\infty} c_n \int_{-\infty}^{\infty} \delta(\omega - n) e^{i\omega x} d\omega = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$