

## Nonlinear Wave propagation

- some examples where there is a NL source + propagation doesn't matter:
  - NL scattering, e.g. isolated quantum dots
  - surface HG, mixing

- most situations involve macroscopic propagation
  - add up NL source over a volume

Wave eqn from Maxwell:

$$\nabla^2 \vec{E} - \frac{\epsilon''}{c^2} \frac{\partial^2}{\partial t^2} \vec{E} = - \frac{1}{c^2 \epsilon_0} \frac{\partial^2}{\partial t^2} \vec{P}^{NL}$$

notes: • assume  $\epsilon'$  is constant over position  
s.t.  $\nabla \cdot \vec{D} = \nabla \cdot \epsilon \vec{E} = \epsilon \nabla \cdot \vec{E} = 0$

- separate linear and NL part of  $\vec{P}$
- if material is birefringent  $\epsilon'' \vec{E} \rightarrow \vec{E} \cdot \vec{E}$
- generally  $E = E(\omega)$  from dispersion

Plane-Wave, nearly CW solutions.

• if  $P^{NL} \ll E \approx E_0 (e^{-i\omega t} + e^{+i\omega t})$

- in perturbative approach  $\rightarrow P^{NL}$  will consist of a sum of  $w$  combinations,

$$\therefore \text{write } \vec{E}(\vec{r}, t) = \sum_{n=0}^{\infty} \left( \vec{E}_n(\vec{r}) e^{-i\omega_n t} + \text{L.C.} \right)$$

similar for  $\vec{P}^{NL}$

$\rightarrow$  set of coupled wave equations

$$\nabla^2 \vec{E}_n(\vec{r}) + \frac{\omega_n^2 \epsilon''(\omega_n)}{c^2} \vec{E}_n(\vec{r}) = - \frac{1}{c^2 \epsilon_0} \frac{\omega_n^2}{c^2} \vec{P}_n^{NL}(\vec{r})$$

$$-\nabla^2 \vec{E}_n(\vec{r}) - \frac{\omega_n^2}{c^2} \epsilon''(\omega_n) \vec{E}_n(\vec{r}) = \frac{1}{\epsilon_0} \frac{\omega_n^2}{c^2} \vec{P}_n^{NL}(\vec{r})$$

• If material is birefringent,  $\epsilon'' \rightarrow \vec{\epsilon}'' = \text{tensor}$   
 $\rightarrow \vec{\epsilon} \cdot \vec{E}_n$

• NL wave eqn is actually 3 eqns for each  $\omega_n$   
 b/c of vector  $\vec{E}$  and  $\vec{P}^{NL}$

• eqns for a  $\omega_n$  couple to eqns for other  $\omega_m$  thru  $\vec{P}^{NL}$   
 eg  $P_i^{(2)} = \epsilon_0 \chi^{(2)} E_m E_n^*$

Second harmonic generation. (simple version)  $\omega_2 = 2\omega_1$   
 Assume for simplicity (bad assumption):

- all waves are plane waves  $\vec{k} = k \hat{z}$

- CW, initially only  $E_1$ , no  $E_2$

- all waves polarized in  $\hat{x}$

$$P^{(2)} = \epsilon_0 \chi^{(2)} E^2 e^{i(k_1 z - \omega_1 t)} e^{-i(k_2 z - \omega_2 t)} e^{i(k_2 z - \omega_2 t)} e^{-i(k_2 z - \omega_2 t)}$$

$$= \epsilon_0 \chi^{(2)} (A_1 e^{i(k_1 z - \omega_1 t)} + A_1^* e^{-i(k_1 z - \omega_1 t)} + A_2 e^{i(k_2 z - \omega_2 t)} + A_2^* e^{-i(k_2 z - \omega_2 t)})$$

define

$$d = \frac{1}{2} \chi^{(2)}$$

eqn for  $\omega_1$ :

$$\frac{\partial^2}{\partial z^2} E_1 + \frac{\omega_1^2}{c^2} \epsilon(\omega_1) E_1 = -2 \frac{\omega_1^2}{c^2} \chi^{(2)} A_1^* A_2 e^{i[(k_2 - k_1)z - (\omega_2 - \omega_1)t]} \quad (2 \text{ b/c of cross terms})$$

As  $E_1, E_2$  propagate, their amplitudes change.  $A_i(z)$

e.g.  $i(k_1 z - \omega_1 t)$

$$E_1(z, t) = A_1(z) e^{i(k_1 z - \omega_1 t)}$$

$$\partial_z^2 E_1 = (A_1' + i k_1 A_1) e^{i(k_1 z - \omega_1 t)}$$

$$\partial_z^2 E_1 = (A_1'' + 2 i k_1 A_1' - k_1^2) e^{i(k_1 z - \omega_1 t)}$$

$$\text{note } k_1^2 = \epsilon''(\omega_1) \frac{\omega_1^2}{c^2}$$

$$\rightarrow (A_1'' + 2 i k_1 A_1') e^{i(k_1 z - \omega_1 t)} = -2 \frac{\omega_1^2}{c^2} A_1^* A_2 e^{i(-k_1 z + \omega_1 t + k_2 z - \omega_2 t)} A_1^* A_2$$

Collect exponentials on RHS

$$A_1'' + 2 i k_1 A_1' = -2 \frac{\omega_1^2}{c^2} A_1^* A_2 \chi^{(2)} e^{i(k_2 - 2k_1)z}$$

Since  $\omega_2 = 2\omega_1$

Slowly varying envelope approximation:

$A_1(z)$  will change over some scale length  $L_2$

$$\therefore \frac{\partial^2 A_1}{\partial z^2} \sim \frac{A_1}{L_2^2}$$

now compare magnitudes of terms on LHS

$$A_1'' + 2ik_1 A_1' \sim \frac{A_1}{L_2^2}, k_1 \frac{A_1}{L_2}$$

if  $k_1 \gg 1/L_2$ , we can drop  $A_1''$

$$\rightarrow \frac{\partial A_1}{\partial z} = \frac{i w_1^2 \chi^{(1)} A_1^* A_2 e^{-i \Delta k z}}{c^2 k_1} \quad (\Delta k = 2k_1 - k_2) \quad \text{"phase mismatch"}$$

go thru same procedure for  $E_2$ :

$$\frac{\partial A_2}{\partial z} = \frac{i w_2^2 \chi^{(2)} A_1^2 e^{+i \Delta k z}}{2 c^2 k_2} \quad (\text{no extra 2 b/c no cross terms})$$

So we have two coupled equations. How to solve?

Try more approximations:

• non-depleted pump  $A_1 = \text{constant}$ .

•  $\Delta k = 0$

$$\rightarrow A_2(z) = \frac{i w_2^2 \chi^{(2)} A_1^2 z}{2 c^2 k_2} = \frac{i w_2 d |A|^2}{c n_1} z$$

calculate intensity:  $\langle \rangle = \text{cycle avg}$

$$\langle |I| \rangle = \langle \epsilon_0 n c |E(t)|^2 \rangle = \langle \epsilon_0 n c |A e^{-i \omega t} + A^* e^{+i \omega t}|^2 \rangle$$

$$\text{let } A = a e^{i \phi} \quad a = \sqrt{\alpha^2 + 2i(\omega t - \phi)} \quad \alpha = \frac{2i(\omega t - \phi)}{c}$$

$$I = \langle \epsilon_0 n c (2a^2 + a^2 e^{-2i(\omega t - \phi)} + a^2 e^{+2i(\omega t - \phi)}) \rangle$$

$$= \epsilon_0 n c \frac{1}{T} \int_0^T (2a^2 + 2a^2 \cos(2\omega t - 2\phi)) dt$$

$$I = 2 \epsilon_0 n c a^2 = 2 \epsilon_0 n c |A|^2$$

compare to

$$\langle I \rangle = \frac{1}{T} \epsilon_0 n c |E|^2 \quad \text{factor of 4 from convention!}$$

$$A_2(z) = i \frac{\omega_2^2 \chi^{(2)}}{2c^2 k_2} A_1^2 z = i \frac{\omega_2^2 d}{c^2 k_2} |A_1|^2 z$$

$$\begin{aligned} I_2(z) &= 2\epsilon_0 c n_2 |A_2(z)|^2 z^2 \\ &= 2\epsilon_0 c n_2 \frac{\omega_2^4 d^2}{c^4 k_2^2} \frac{I_1^2}{(2\epsilon_0 c n_1)^2} z^2 \\ &= \frac{1}{2\epsilon_0 n_1^2 n_2 c^3} \frac{d^2 \omega_2^2}{k_2^2} I_1^2 z^2 \end{aligned}$$

unit check?

$$P = \epsilon_0 \chi^{(2)} E^2 \quad \left. \right\} \text{same units}$$

$$P^{(1)} = \epsilon_0 \chi^{(1)} E$$

$$E = 1 + \chi^{(1)} \quad \therefore \chi^{(1)} \text{ is dimensionless.}$$

$$\therefore \chi^{(2)} E \text{ is dimensionless}$$

$$E \approx V/m \quad \chi^{(2)} \approx 4 \text{ pm/V typically.}$$

$$\text{since } d^2 = \left(\frac{1}{2} \chi^{(2)}\right)^2 \text{ then } \frac{d^2}{2\epsilon_0 c n} \approx \frac{1}{\text{intensity.}}$$

$$I_2 \sim \frac{d^2}{c} I_1^2 \left(\frac{\omega_2 z}{c}\right)^2 \sim I_1 \quad \checkmark$$

$$\frac{\epsilon_0 c}{d^2} \approx 1.66 \times 10^{-16} \text{ W/cm}^2$$

$$\text{let } n_1 = n_2 = 1.5$$

$$\lambda_2 = 500 \text{ nm} \rightarrow \omega_2 = 3.77 \times 10^{15} \text{ rad/s}$$

$$z = 10 \text{ mm}$$

$$\text{when } I_1 \approx 3.3 \times 10^8 \text{ W/cm}^2 \quad I_2(z) \approx I_1$$

$$\text{if } w_0 = 0.7 \text{ mm} \rightarrow P = 10.5 \text{ W}$$

Phase matching:

go back to

$$\frac{\partial A_2}{\partial z} = i \frac{w_2^2 d}{k_2 c^2} A_1^2 e^{i \Delta k z} \quad \text{for } \Delta k \neq 0$$

$$\Delta k \equiv 2k_1 - k_2 = 2 \frac{n_1 w_1}{c} - \frac{n_2 w_2}{c}$$

but since  $w_2 = 2w_1$ ,

$$\Delta k = \frac{w_2}{c} (n_1 - n_2)$$

for transparent materials  $n(\omega)$  increases with  $\omega$



$\rightarrow \Delta k < 0$  typically.

Integrate:

$$\begin{aligned} A_2(z) &= i \frac{w_2^2 d}{k_2 c^2} A_1^2 \left\{ e^{i \Delta k z} dz \right. \\ &= " \left( \frac{e^{i \Delta k L}}{i \Delta k} - 1 \right) = e^{-i \frac{\Delta k L}{2}} \frac{(e^{i \frac{\Delta k L}{2}} - e^{-i \frac{\Delta k L}{2}})}{i \Delta k} \\ &= L e^{i \frac{\Delta k L}{2}} \frac{\sin(\frac{\Delta k L}{2})}{\Delta k L / 2} = L e^{i \frac{\Delta k L}{2}} \operatorname{sinc}\left(\frac{\Delta k L}{2}\right) \end{aligned}$$

$$\rightarrow I_2(z) = \frac{1}{2} \frac{d^2 w_2^2}{E_0 n_1^2 n_2 c^3} I_1^2 L^2 \operatorname{sinc}^2\left(\frac{\Delta k L}{2}\right)$$