

Gaussian beams and ABCD

Gaussian beam solution to EM wave equation

Complex q formulation

“Standard” form of Gaussian beam equations

High-order modes

Gaussian beams and ABCD

examples of Gaussian beam propagation

Gaussian beam solution to wave equation

- Use Fresnel integral to propagate a Gaussian beam

$$u(x, y, z) = \frac{i}{\lambda L} e^{-ik \frac{x^2 + y^2}{2L}} \iint e^{-\frac{x'^2 + y'^2}{w^2}} e^{-ik \frac{x'^2 + y'^2}{2L}} e^{+i(\beta_x x' + \beta_y y')} dx' dy'$$

- Combine quadratic terms in exponent:

$$\left(\frac{1}{w^2} + i \frac{k}{2L} \right) = i \frac{k}{2} \left(\frac{1}{L} - i \frac{2}{kw^2} \right) = i \frac{k}{2q}$$

- Now integral is a F.T. of a complex Gaussian=Gaussian

$$u(x, y, z) = \frac{i}{\lambda L} e^{-ik \frac{x^2 + y^2}{2L}} \iint e^{-ik \frac{x'^2 + y'^2}{2q}} e^{+i(\beta_x x' + \beta_y y')} dx' dy'$$

$$\rightarrow u(r, z) = \frac{1}{q(z)} e^{-ik \frac{r^2}{2q(z)}} \quad \frac{1}{q(z)} = \frac{1}{R(z)} - i \frac{\lambda}{\pi w^2(z)}$$

Complex q form for Gaussian beam

- This combines beam size and radius of curvature into one complex parameter
 - This form is used for ABCD calculations

$$A(r, z) = A_0 \frac{1}{1 + i\xi} e^{-\frac{r^2}{w_0^2(1+i\xi)}} \quad \rightarrow \quad A(r, z) = \frac{1}{q(z)} e^{-ik\frac{r^2}{2q(z)}}$$

$$\frac{1}{q(z)} = \frac{1}{R(z)} - i \frac{\lambda}{\pi w^2(z)}$$

$$\begin{aligned} \frac{1}{q(z)} &= \frac{1}{z + iz_R} = \frac{z}{z^2 + z_R^2} - i \frac{z_R}{z^2 + z_R^2} \\ &= \frac{1}{R(z)} - i \frac{w_0^2}{z_R w^2(z)} \\ &= \frac{1}{R(z)} - i \frac{\lambda}{\pi w^2(z)} = \frac{1}{R(z)} - i \frac{1}{Z(z)} \end{aligned}$$

$$\frac{1}{R(z)} = \frac{1}{z \left(1 + \frac{z_R^2}{z^2} \right)} = \frac{z}{z^2 + z_R^2}$$

$$\frac{1}{w^2(z)} = \frac{1}{w_0^2 \left(1 + \frac{z^2}{z_R^2} \right)} = \frac{z_R^2}{w_0^2 (z^2 + z_R^2)}$$

Complex q vs standard form

$$u(r, z) = \frac{1}{q(z)} e^{-ik \frac{r^2}{2q(z)}} \quad \text{with} \quad \frac{1}{q(z)} = \frac{1}{R(z)} - i \frac{\lambda}{\pi w^2(z)}$$

Expand exponential:

$$\begin{aligned} \exp\left[-ik \frac{r^2}{2q(z)}\right] &= \exp\left[-ik \frac{r^2}{2} \left(\frac{1}{R(z)} - i \frac{\lambda}{\pi w^2(z)}\right)\right] \\ &= \exp\left[-ik \frac{r^2}{2} \frac{1}{R(z)} - i \frac{2\pi r^2}{\lambda} \frac{1}{2} \left(-i \frac{\lambda}{\pi w^2(z)}\right)\right] = e^{-ik \frac{r^2}{2R(z)}} e^{-\frac{r^2}{w^2(z)}} \end{aligned}$$

$$a + ib = \sqrt{a^2 + b^2} e^{i \arctan(b/a)}$$

Expand leading inverse q:

$$\begin{aligned} \frac{1}{q(z)} &= -i \left(\frac{iz}{z^2 + z_R^2} + \frac{z_R}{z^2 + z_R^2} \right) = -i \left(\frac{\sqrt{z^2 + z_R^2}}{z^2 + z_R^2} \right) e^{i \arctan(z/z_R)} \\ &= -i \left(\frac{1}{z_R \sqrt{1 + z^2/z_R^2}} \right) e^{i \arctan(z/z_R)} = \frac{w_0}{i z_R w(z)} e^{i \eta(z)} \end{aligned}$$

Standard form of Gaussian beam solutions

$$E(r, z, t) = A_0 e^{-i(kz - \omega t)} \frac{w_0}{w(z)} e^{-\frac{r^2}{w^2(z)}} e^{-i\frac{kr^2}{2R(z)}} e^{i\eta(z)}$$

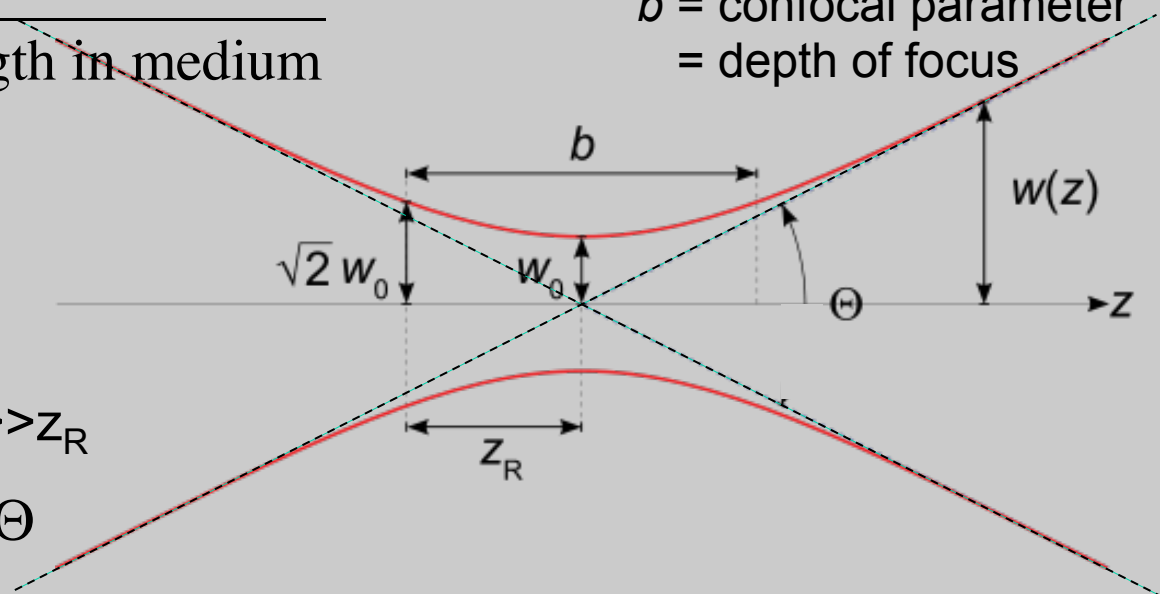
Beam maintains a Gaussian profile as it propagates

- beam radius that varies with z
- Origin of z coordinate is at the beam waist
- Rayleigh length z_R defines collimation distance from focal plane

$$z_R = \frac{\pi w_0^2}{\lambda / n} = \frac{\text{beam area}}{\text{wavelength in medium}}$$

b = confocal parameter
= depth of focus

$$w(z) = w_0 \sqrt{1 + \frac{z^2}{z_R^2}}$$



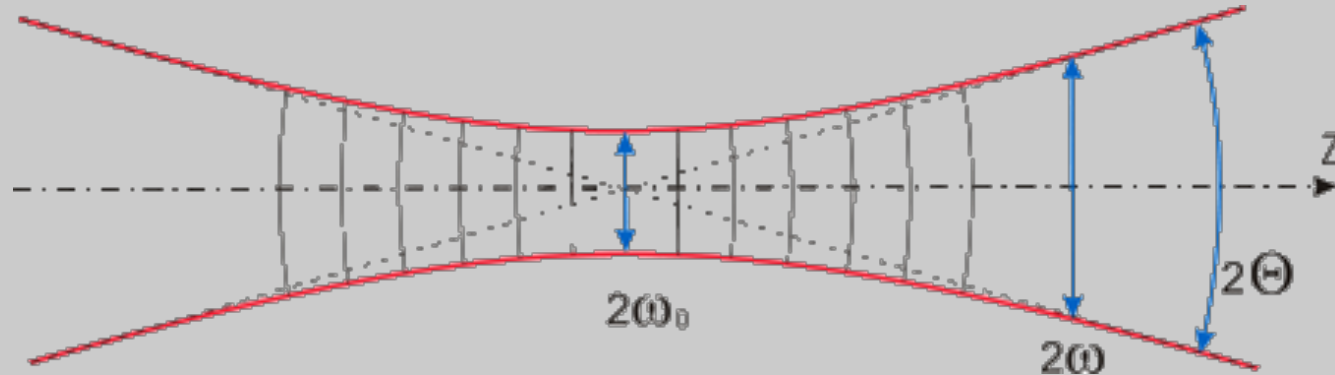
Geometric limit for $z \gg z_R$

$$w(z) = z \frac{w_0}{z_R} = z \tan \Theta$$

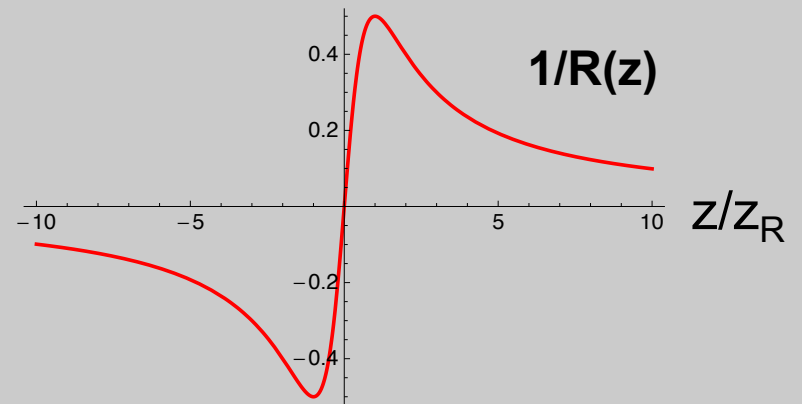
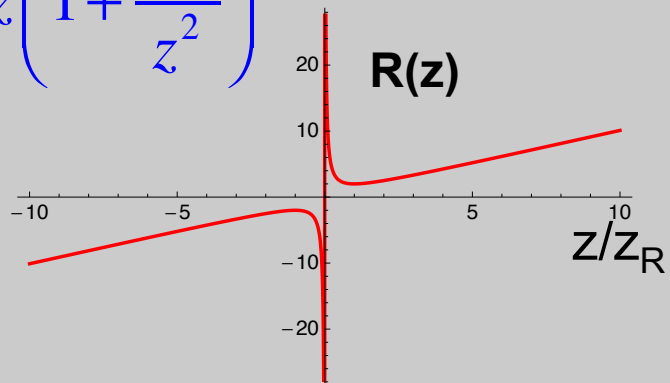
Evolution of wavefronts

$$E(r, z, t) = A_0 e^{-i(kz - \omega t)} \frac{w_0}{w(z)} e^{-\frac{r^2}{w^2(z)}} e^{-i\frac{kr^2}{2R(z)}} e^{i\eta(z)}$$

- Wavefront curvature evolves with z as beam size changes



$$R(z) = z \left(1 + \frac{z_R^2}{z^2} \right)$$



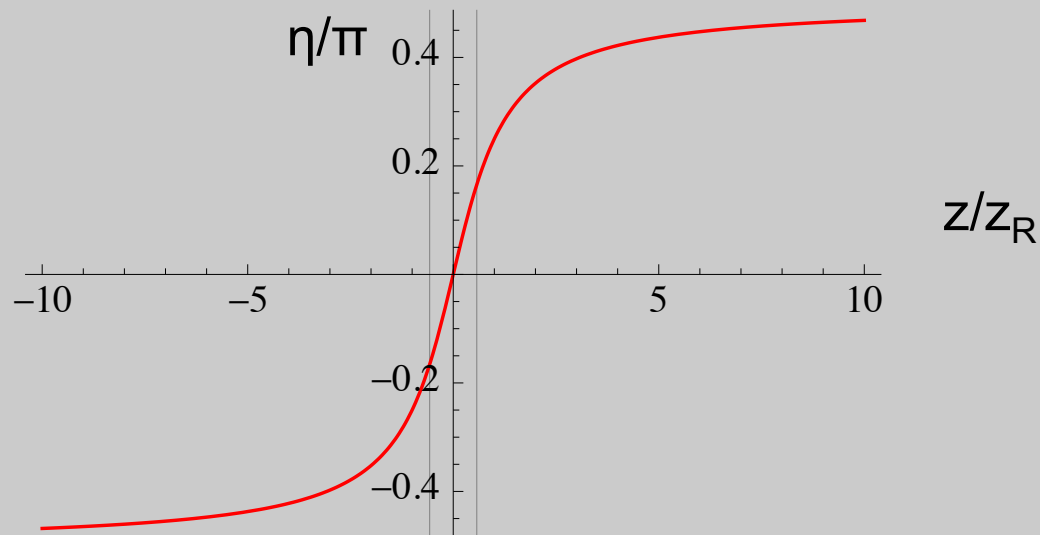
On-axis phase: Gouy phase

$$E(r, z, t) = A_0 e^{-i(kz - \eta(z) - \omega t)} \frac{W_0}{w(z)} e^{-\frac{r^2}{w^2(z)}} e^{-i\frac{kr^2}{2R(z)}}$$

- Because the wavefront changes from focusing to defocusing, on-axis phase advances with z

Gouy phase

$$\eta(z) = \arctan\left(\frac{z}{z_R}\right)$$



Higher-order Hermite-Gauss modes

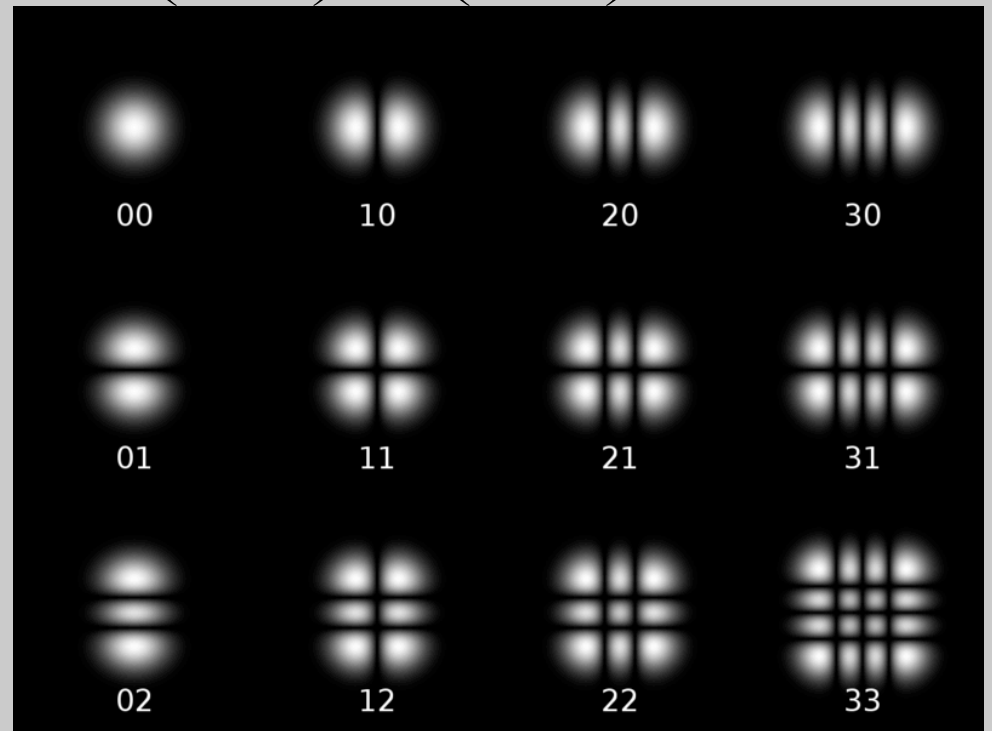
- The Gaussian beam is just the lowest order mode solution to the wave equation
- x, y coordinates: Hermite-Gaussian modes

$$E(x, y, z) = A_0 e^{-i(kz - \eta_{lm}(z))} \frac{w_0}{w(z)} e^{-\frac{x^2 + y^2}{w^2(z)}} H_l \left(\frac{\sqrt{2}x}{w(z)} \right) H_m \left(\frac{\sqrt{2}y}{w(z)} \right) e^{-i\frac{k(x^2 + y^2)}{2R(z)}}$$

$$\eta_{lm} = (l + m + 1) \tan^{-1} \left(\frac{z}{z_R} \right)$$

Transverse profile is maintained during propagation (scaled with $w(z)$)

Hermite-Gauss functions are the same as solutions to quantum SHO

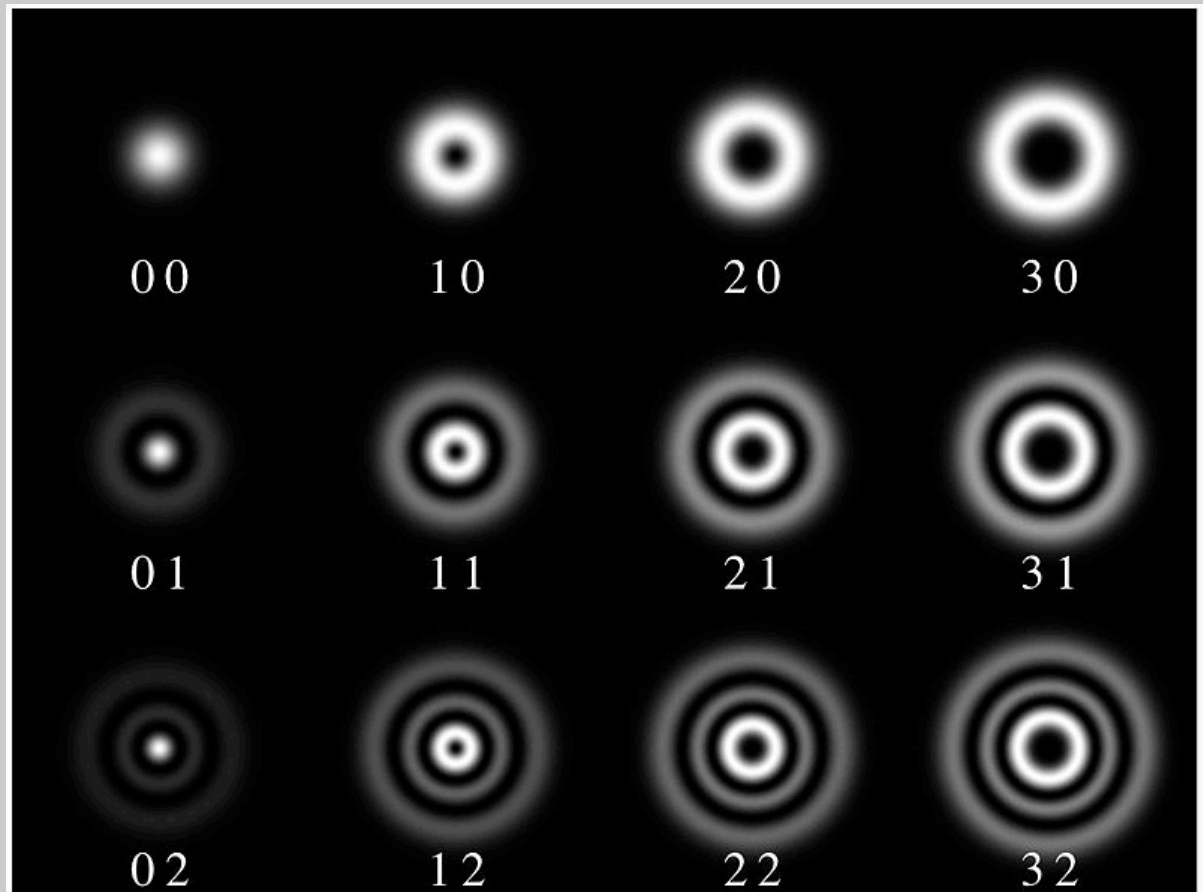


Higher-order LaGuerre-Gauss modes

- In cylindrical coordinates, alternate representation
- Azimuthal phase $\exp[im\phi]$ “vortex” phase

Example:

LG10 mode is a linear combination of HG10 and HG01



Difference between Siegman's complex q and standard form

$$u(r, z) = \frac{1}{q(z)} e^{-ik \frac{r^2}{2q(z)}} = \frac{1}{i z_R} \frac{w_0}{w(z)} e^{i\eta(z)} e^{-ik \frac{r^2}{2R(z)}} e^{-\frac{r^2}{w^2(z)}}$$

$$E(r, z, t) = A_0 \frac{w_0}{w(z)} e^{i(kz - \omega t)} e^{-\frac{r^2}{w^2(z)}} e^{i \frac{kr^2}{2R(z)}} e^{-i\eta(z)}$$

- Siegman's form for the complex q is used almost everywhere for the ABCD calculations.
- He uses the $\exp[+ i w t]$ convention, which accounts for the sign difference in the complex exponentials.
- With $\exp[-i w t]$ convention, define q as:

$$\frac{1}{q(z)} = \frac{1}{R(z)} + i \frac{\lambda}{\pi w^2(z)} = \frac{1}{z - i z_R}$$

Compare Boyd's form to standard:

- Boyd's complex form is consistent with standard Gaussian beam form

$$A(r, z) = A_0 \frac{1}{1 + i\xi} e^{-\frac{r^2}{w_0^2(1+i\xi)}} = A_0 \frac{1}{1 + iz/z_R} e^{-\frac{r^2}{w_0^2(1+iz/z_R)}}$$

$$\frac{1}{1 + i\xi} = \frac{1}{1 + iz/z_R} = \frac{z_R}{z_R + iz} = \frac{-iz_R}{z - iz_R} = \frac{-iz_R}{q(z)}$$

$$A(r, z) = A_0 (-iz_R) \frac{1}{q(z)} e^{+\frac{iz_R r^2}{w_0^2 q(z)}} = -iz_R A_0 \frac{1}{q(z)} e^{+\frac{ikr^2}{2q(z)}}$$

Gaussian beams and ABCD

- General expression

$$q_1 = \frac{Aq_0 + B}{Cq_0 + D} \qquad \frac{1}{q(z)} = \frac{1}{R(z)} - i \frac{\lambda}{\pi w^2(z)}$$

- Since q is defined through its inverse, alternate:

$$q_1^{-1} = \frac{C + Dq_0^{-1}}{A + Bq_0^{-1}}$$

- Note that ABCD matrices are the same as for raytrace
- Application is **not** a multiplication like matrix.vector

Simple Gaussian ABCD examples: translation

- translation

$$\frac{1}{q(z)} = \frac{1}{R(z)} - i \frac{\lambda}{\pi w^2(z)} = \frac{1}{z \left(1 + \frac{z_R^2}{z^2}\right)} - i \frac{\lambda}{\pi w_0^2 \left(1 + \frac{z^2}{z_R^2}\right)}$$

$$R(z) = z \left(1 + \frac{z_R^2}{z^2}\right)$$

$$w(z) = w_0 \sqrt{1 + \frac{z^2}{z_R^2}}$$

$$= \frac{1}{z_R \left(1 + \frac{z^2}{z_R^2}\right)} (z/z_R - i)$$

$$q(z) = z_R \left(1 + \frac{z^2}{z_R^2}\right) \frac{1}{(z/z_R - i)}$$

$$= z_R \left(1 + \frac{z^2}{z_R^2}\right) \frac{z/z_R + i}{\left(1 + \frac{z^2}{z_R^2}\right)} = z + iz_R$$

$$q(z) = z + iz_R \rightarrow q_1 = z_0 + L + iz_R$$

ABCD for translation:

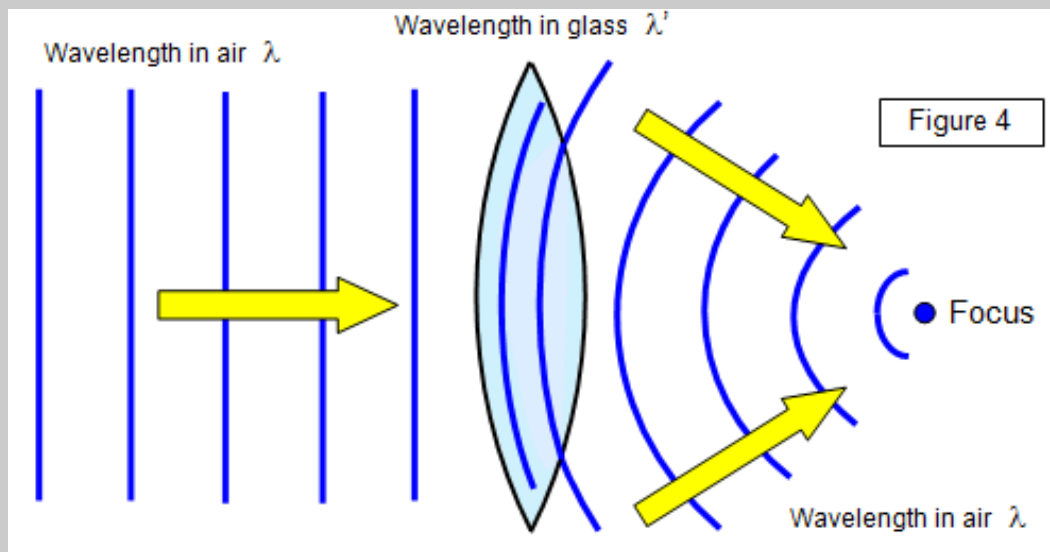
$$\begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$$

$$q_1 = \frac{Aq_0 + B}{Cq_0 + D} = q_0 + L$$

Simple Gaussian ABCD examples: lens

- Focusing by a lens

- Radius of curvature is modified by lens: $\frac{1}{R'} = \frac{1}{R} - \frac{1}{f}$



Focusing by lens induces a negative ROC

$$\frac{1}{q_1} = \frac{1}{R} - \frac{1}{f} - i \frac{\lambda}{\pi w^2} = \frac{1}{q_0} - \frac{1}{f}$$

$$q_1^{-1} = \frac{C + Dq_0^{-1}}{A + Bq_0^{-1}} \quad \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix}$$

Focusing a Gaussian beam by a lens

- For a beam waist at lens entrance, distance from lens to focused waist is not exactly = f
- Define variables:

w_{01} (w_{02}) = input (focused) beam waist radius

z_{R1} (z_{R2}) = rayleigh range for input (focused) beam

z_m = distance from lens to focused beam waist

- Use Gaussian beam equations to back propagate to lens

$$w_{01} = w(z = -z_m) = w_{02} \sqrt{1 + \frac{z_m^2}{z_{R2}^2}} \quad \rightarrow \quad z_{R1} = \frac{\pi w_{01}^2}{\lambda} = z_{R2} \left(1 + \frac{z_m^2}{z_{R2}^2} \right)$$

$$R(z = -z_m) = -f = -z_m \left(1 + \frac{z_{R2}^2}{z_m^2} \right)$$

$$\frac{f}{z_{R1}} = \frac{z_m \left(1 + \frac{z_{R2}^2}{z_m^2} \right)}{z_{R2} \left(1 + \frac{z_m^2}{z_{R2}^2} \right)}$$

- Divide equations:

Gaussian beam focusing

- From prev slide:

$$\frac{f}{z_{R1}} = \frac{z_m \left(1 + \frac{z_{R2}^2}{z_m^2} \right)}{z_{R2} \left(1 + \frac{z_m^2}{z_{R2}^2} \right)}$$

- Let $x_1 = \frac{f}{z_{R1}}$ $x_2 = \frac{z_{R2}}{z_m}$

$$\rightarrow x_1 = \frac{1}{x_2} \frac{1 + x_2^2}{1 + \frac{1}{x_2^2}} = \frac{1}{x_2} x_2^2 \frac{1 + x_2^2}{1 + x_2^2} = x_2 \quad \frac{f}{z_{R1}} = \frac{z_{R2}}{z_m}$$

- Go back to expressions for z_{R1} and f :

$$z_{R1} = z_{R2} \left(1 + \frac{z_m^2}{z_{R2}^2} \right) = z_{R2} \left(1 + \frac{z_{R1}^2}{f^2} \right) \quad \rightarrow z_{R2} = \frac{z_{R1}}{1 + \frac{z_{R1}^2}{f^2}}$$

$$f = z_m \left(1 + \frac{z_{R2}^2}{z_m^2} \right) = z_m \left(1 + \frac{f^2}{z_{R1}^2} \right) \quad \rightarrow z_m = \frac{f}{1 + \frac{f^2}{z_{R1}^2}}$$

Key parameter is the ratio: $\frac{f}{z_{R1}}$

Interpretation of results

- Ratio f/z_{R1} determines effect of input beam Rayleigh range on position of focus

- Distance to beam waist is *shorter* than f

- This matters only when z_{R1} is comparable to f

$$z_m = \frac{f}{1 + \frac{f^2}{z_{R1}^2}}$$

- Focused spot size:

- For $f/z_{R1} \ll 1$,

$$z_{R2} = \frac{z_{R1}}{1 + \frac{z_{R1}^2}{f^2}} \rightarrow w_{02} = \frac{w_{01}}{\sqrt{1 + \frac{z_{R1}^2}{f^2}}}$$

$$w_{02} = \frac{w_{01}}{\frac{z_{R1}}{f} \sqrt{1 + \frac{f^2}{z_{R1}^2}}} \approx \frac{f w_{01}}{z_{R1}} = \frac{\lambda f}{\pi w_{01}}$$

$$w_{02} = \frac{2}{\pi} \lambda \frac{f}{2w_{01}} = \frac{2}{\pi} \lambda F / \#$$

F-number of focusing for a beam is defined as $f/\text{beam dia}$

Focused Gaussian beam: ABCD version

- Define input q :
$$\frac{1}{q_0} = \frac{1}{R_0} - i \frac{\lambda}{\pi w_{01}^2} = -i \frac{1}{z_{R1}}$$

- Lens: $\begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix}$ q just after lens: $q_1^{-1} = \frac{C + Dq_0^{-1}}{A + Bq_0^{-1}}$

$$\frac{1}{q_1} = \frac{1}{q_0} - \frac{1}{f} = -\frac{1}{f} - i \frac{1}{z_{R1}}$$

- Translation by z_m : $\begin{pmatrix} 1 & z_m \\ 0 & 1 \end{pmatrix}$ $q_2^{-1} = \frac{C + Dq_1^{-1}}{A + Bq_1^{-1}} = \frac{-\frac{1}{f} - i \frac{1}{z_{R1}}}{1 + z_m \left(-\frac{1}{f} - i \frac{1}{z_{R1}} \right)}$

- At z_m , beam is at a waist, so

$$\frac{1}{q_2} = -i \frac{\lambda}{\pi w_{01}^2} = -i \frac{1}{z_{R2}}$$

Then solve.