Reading : G 8.1 (Review 7.3)

Tomorrow : G. 8.2

Work and energy

From Phys I: $W = \int \vec{F} \cdot d\vec{l} = \Delta\left(\frac{1}{2}mv^2\right)$

From Newton's 2nd Law for a particle

$$m\frac{d\vec{v}}{dt} = F$$

Charge conservation

$$\vec{\nabla} \cdot \vec{J} = -\frac{\partial \rho}{\partial t}$$

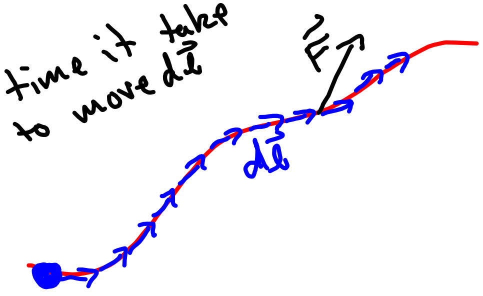
$$\int \vec{\nabla} \cdot \vec{J}\, dV = -\int \frac{\partial \rho}{\partial t} dV \Rightarrow \int \vec{J} \cdot d\vec{a} = -\frac{\partial q_{enc}}{\partial t}$$

What about energy in E&M

$W = \int \vec{F}\cdot d\vec{l}$

$= \int \vec{F}\cdot d\vec{l}\,\frac{dt}{dt}$ ← time it take to move $d\vec{l}$

$= \int \vec{F}\cdot \frac{d\vec{l}}{dt}\,dt$



$W = \int \vec{F}\cdot\vec{v}\,dt = \int \frac{dW}{dt}\,dt = \int \frac{dU_{mech}}{dt}\,dt$

↑ Power (energy delivered per time)

For E&M, what's $\vec{F}$? $\vec{F} = q\vec{E} + q\vec{v}\times\vec{B}$

Power being delivered is:

$\vec{F}\cdot\vec{v} = q(\vec{E}\cdot\vec{v}) + \cancel{(q\vec{v}\times\vec{B})\cdot\vec{v}} \rightarrow \phi$

What is q in volume?

$q = \int \rho\, dV = \int dq$

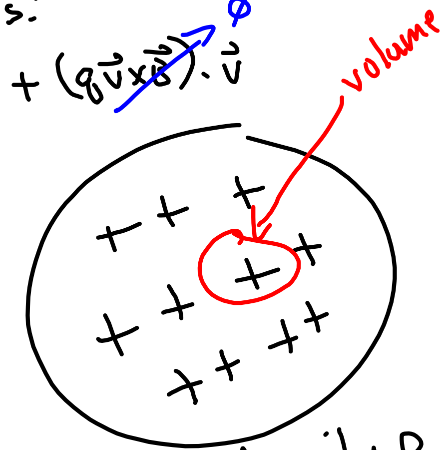
Take limit as volume goes to small.

$dq \approx \rho\, dV$

Power being delivered to dq

$\vec{F}\cdot\vec{v} = \rho\, dV\,(\vec{E}\cdot\vec{v})$

$= (\vec{E}\cdot\vec{J})\,dV \quad \{\vec{J} = \rho\vec{v}\}$



Charge density ρ

Let's look at $\vec{E}\cdot\vec{J}$, $\vec{J}\cdot\vec{E}$.

Use ME: $\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0 \frac{\partial \vec{E}}{\partial t}$

$1/c^2$

$$\Rightarrow \vec{J} = \frac{1}{\mu_0}\vec{\nabla}\times\vec{B} - \cancel{\mu_0}\epsilon_0\frac{\partial\vec{E}}{\partial t}$$

$$\vec{J}\cdot\vec{E} = \left(\frac{1}{\mu_0}\vec{\nabla}\times\vec{B} - \cancel{\mu_0}\epsilon_0\frac{\partial\vec{E}}{\partial t}\right)\cdot\vec{E}$$

$$= \frac{1}{\mu_0}(\vec{\nabla}\times\vec{B})\cdot\vec{E} - \cancel{\mu_0}\epsilon_0\frac{\partial\vec{E}}{\partial t}\cdot\vec{E}$$

Vector identity:

$$\vec{\nabla}\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\vec{\nabla}\times\vec{A}) - \vec{A}\cdot(\vec{\nabla}\times\vec{B})$$

$\Rightarrow \vec{A}\to\vec{E}$, $\vec{B}\to\vec{B}$

$$\vec{E}\cdot(\vec{\nabla}\times\vec{B}) = \vec{B}\cdot(\vec{\nabla}\times\vec{E}) - \vec{\nabla}\cdot(\vec{E}\times\vec{B})$$

$$= \frac{1}{\mu_0}\left[\vec{B}\cdot(\vec{\nabla}\times\vec{E}) - \vec{\nabla}\cdot(\vec{E}\times\vec{B})\right] - \epsilon_0\frac{\partial\vec{E}}{\partial t}\cdot\vec{E}$$

Using ME: $\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$

$$\vec{E}\cdot\vec{J} = \frac{1}{\mu_0}\left[\vec{B}\cdot\left(-\frac{\partial\vec{B}}{\partial t}\right) - \vec{\nabla}\cdot(\vec{E}\times\vec{B})\right] - \epsilon_0\frac{\partial\vec{E}}{\partial t}\cdot\vec{E}$$

$$= -\frac{1}{\mu_0}\vec{B}\cdot\frac{\partial\vec{B}}{\partial t} - \epsilon_0\vec{E}\cdot\frac{\partial\vec{E}}{\partial t} - \frac{1}{\mu_0}\vec{\nabla}\cdot(\vec{E}\times\vec{B})$$

$\frac{\partial}{\partial t}(B^2) = 2\vec{B}\cdot\frac{\partial\vec{B}}{\partial t}$; for $\vec{E}$ as well

$$= -\frac{\partial}{\partial t}\underbrace{\left(\frac{1}{2\mu_0}B^2 + \frac{\epsilon_0}{2}E^2\right)}_{u_{em}} - \vec{\nabla}\cdot\underbrace{\left(\frac{1}{\mu_0}\vec{E}\times\vec{B}\right)}_{\vec{S}}$$

{EM energy dens} {Poynting Vector}

$$\vec{J}\cdot\vec{E} = -\frac{\partial u_{em}}{\partial t} - \vec{\nabla}\cdot\vec{S}$$

$$-\frac{\partial u_{em}}{\partial t} = \vec{\nabla}\cdot\vec{S} + \vec{J}\cdot\vec{E}$$

Conservation Law for energy in E&M.

$$u_{em} = \frac{1}{2\mu_0} B^2 + \frac{\epsilon_0}{2} E^2$$

To find U_{em} {total energy} in a volume

$$U_{em} = \int \frac{1}{2\mu_0} B^2 + \frac{\epsilon_0}{2} E^2 \, dV$$

Conservation law in integral form:

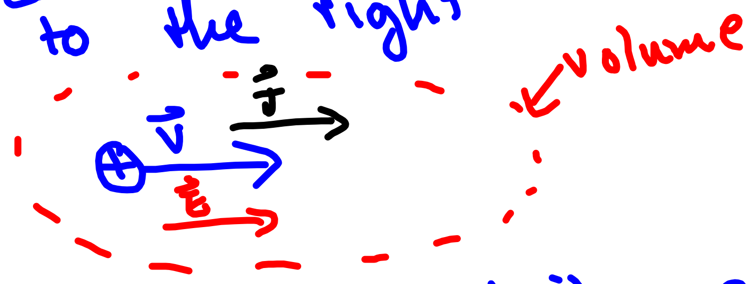
$$-\frac{\partial}{\partial t}\left(\int u_{em} \, dV\right) = \oint \vec{S} \cdot d\vec{a} + \int \vec{J} \cdot \vec{E} \, dV$$

Power being delivered to the charges inside the volume.

$$-\frac{\partial}{\partial t} U_{em} - \underbrace{\frac{\partial}{\partial t} U_{mech}}_{\int \vec{J} \cdot E \, dV} = \oint \vec{S} \cdot d\vec{a}$$

Let's talk about signs

Lets say you have a positive charge moving to the right



If I want to speed it up, which direction should $\vec{E}$ point? To the right.

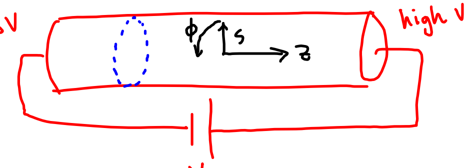
$$\vec{J}\cdot\vec{E} > \emptyset$$

Take the volume out to infinity

$$\oint \vec{S}\cdot d\vec{a} = \emptyset$$

$$\rightarrow \quad \int \vec{J}\cdot\vec{E}\, dV = -\frac{\partial}{\partial t}\int u_{em}\, dV$$

You have a wire of radius R, length L.



You apply a voltage V to it, and current flows. Let's assume constant resistivity, this leads to a constant E-field along wire.

$$\Delta V = -\int \vec{E}\cdot d\vec{l} \Rightarrow \underbrace{\vec{E} = -\vec{\nabla} V}_{\text{only for statics}}$$

$$V = EL$$

$$\vec{E} = -\frac{V}{L}\hat{z}$$

What is $\vec{B}$ at the surface of the wire given a total current I is flowing?

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enc} + \mu_0\epsilon_0 \cancel{\frac{d}{dt}\int \vec{E}\cdot d\vec{a}}^{\,0}$$

$$\int B_\phi R\, d\phi = \mu_0 I$$

$$B_\phi R \int d\phi = 2\pi B_\phi R = \mu_0 I$$

$$B_\phi = \frac{\mu_0 I}{2\pi R} \Rightarrow \vec{B} = -\frac{\mu_0 I}{2\pi R}\hat{\phi}$$

What is EM power flow through the surface of the wire?

EM Power flow: $\int \vec{S}\cdot d\vec{a}$

$$\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B} = \frac{1}{\mu_0}\left(-\frac{V}{L}\hat{z}\right)\times\left(-\frac{\mu_0 I}{2\pi R}\hat{\phi}\right)$$

$$= -\frac{VI}{2\pi RL}\hat{s}$$

$$\int \vec{S}\cdot d\vec{a} = \int \vec{S}\cdot dz\, s\, d\phi\, \hat{s}$$

$$= \int_0^{2\pi}\int_0^{L} \frac{-VI}{2\pi RL} R\, dz\, d\phi$$

$$= -IV$$