Note Title 3/6/2008

New notation for inner products

for finite dimensional vectors:

Let $|\alpha\rangle$ be an N-dim. vector

Let $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{pmatrix}$ be the comp. of this vector wrt a particular basis

The inner product of two vectors $|\alpha\rangle$ and $|\beta\rangle$ is :

$$\langle\alpha|\beta\rangle = a_1^* b_1 + a_2^* b_2 + \cdots a_N^* b_N$$

we would say that $\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{pmatrix}$ represents $|\alpha\rangle$.

Similarly matrices represent Linear operators.

$$|b\rangle = T|\alpha\rangle \to \vec{b} = T\vec{a} = \begin{pmatrix} t_{11} & t_{12} & t_{13} \cdots \\ t_{21} & t_{22} & \\ & & \cdots & t_{NN} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{pmatrix}$$

For the most part, the vectors in QM are functions.

$f(x)$, $g(x)$ then

$$\langle f | g \rangle = \int_a^b f^*(x)\, g(x)\, dx$$

The set of all functions on $[a,b]$ such that $\int_a^b |f(x)|^2 dx < \infty$ is a linear vector space called $\mathcal{L}_2[a,b]$. It is an example of a Hilbert space

Hilbert space: a complete inner product space $\mathcal{H}$ where the inner product defines the norm: $\|f\|^2 = \sqrt{\langle f|f \rangle}$

Completeness means that if $f_n \in \mathcal{H}$ converge to f, then f is in $\mathcal{H}$ too.

Using our new notation for inner products:

Let $Q(x,p)$ be an observable. This is just a function. $\hat{Q}$ is the operator obtained by replacing $p \Rightarrow \hat{p} = -i\hbar \frac{d}{dx}$

$$\underset{\substack{\downarrow \\ \text{expectation}}}{\langle Q \rangle} = \int \psi^* \hat{Q} \psi \, dx = \underbrace{\langle \psi | \hat{Q} \psi \rangle}_{\text{inner product}}$$

$\langle Q \rangle$ is the average of many measurements, so $\langle Q \rangle$ must be real: $\langle Q \rangle = \langle Q \rangle^*$

$$\langle \psi | \hat{Q} \psi \rangle^* = \langle \hat{Q} \psi | \psi \rangle$$
$$= \langle \psi | \hat{Q} \psi \rangle$$

$$\langle f | g \rangle^* = \langle g | f \rangle$$

So observables must be represented by Hermitian operators.

Determinant states

If you measure an observable Q many times, each time for an identically prepared system in the state ψ, then you do not get the same result each time.

This is called indeterminacy

In contrast is ψ_E is an eigenstate of H (stationary state) with eigenvalue E, then every measurement of energy will yield E. This is an example of a determinant state.

The standard deviation of an observable Q in a determinant state must be zero.

$$\sigma^2 \equiv \langle (\hat{Q} - \langle \hat{Q} \rangle)^2 \rangle = \langle \psi | (\hat{Q} - \overbrace{q}^{\text{a number now}})^2 \psi \rangle$$

$$= \langle \psi | (\hat{Q} - q)(\hat{Q} - q)\psi \rangle$$

$$= \langle (\hat{Q} - q)\psi | (\hat{Q} - q)\psi \rangle = 0$$

if $\hat{Q}$ is Hermitian then so is $\hat{Q} - q$.

So we have $\langle \quad | \quad \rangle = 0$

this implies that $\quad = 0$

so $(\hat{Q} - q)\psi = 0$ or $\boxed{\hat{Q}\psi = q\psi}$

Thus determinant states must be eigenvectors of the operator associated with the observable.

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more eigenfunction examples

Let $\hat{Q} \equiv i \frac{d}{d\phi}$ $\phi \in [0, 2\pi]$

our set of functions will be $f(\phi)$ such that $f(\phi) = f(\phi + 2\pi)$

$\hat{Q} f = \lambda f$ is the E-value equation

$i \frac{d}{d\phi} f = \lambda f$ or $\boxed{f' = -i\lambda f}$

$\Rightarrow f(\phi) = A e^{-i\lambda\phi}$

in order that f be periodic with period 2π

$$A e^{-i\lambda\phi} = A e^{-i\lambda(\phi + 2\pi)} = A e^{-i\lambda\phi} e^{-i\lambda 2\pi}$$

$$\Rightarrow e^{-i\lambda 2\pi} = 1$$

$$\Rightarrow \lambda = 0, \pm 1, \pm 2, \ldots$$

The eigenvalues of an operator are called its spectrum. This operator has a discrete spectrum

$$-i\hbar \frac{d}{dx} f_p(x) = p\, f_p(x)$$

$$\Rightarrow \quad f_p(x) = A\, e^{ipx/\hbar}$$

This works for any p (continuous spectrum), but the resulting eigenfunction is not square integrable.

$$\langle f_p(x) \mid f_p(x) \rangle = \int_{-\infty}^{\infty} A^* e^{-ipx/\hbar} A\, e^{ipx/\hbar}\, dx$$

$$= |A|^2 \int_{-\infty}^{\infty} dx = \infty$$

However consider orthonormality condition

$$\langle f_{p'}(x) \mid f_p(x) \rangle = |A|^2 \int_{-\infty}^{\infty} e^{i(p-p')x/\hbar}\, dx$$

we know that $\int_{-\infty}^{\infty} e^{ipx}\, dx = \delta(p)$

So let $x/\hbar = y$ $\quad \hbar\, dy = dx$

$$\langle f_{p'}(x) | f_p(x) \rangle = |A|^2 \hbar \underbrace{\int_{-\infty}^{\infty} e^{-i(p-p')y}\, dy}_{2\pi\, \delta(p-p')}$$

$$= 2\pi\hbar |A|^2 \delta(p-p')$$

Let $|A|^2 = \frac{1}{\sqrt{2\pi\hbar}}$

have

$$\boxed{\langle f_{p'} | f_p \rangle = \delta(p-p')}$$

which is very like orthogonality

So, the eigenfunctions of $\hat{p}$ are

$$\frac{1}{\sqrt{2\pi\hbar}}\, e^{ipx/\hbar}$$

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are these eigenfunctions complete?

i.e. can any

$$f(x) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} c(p)\, e^{\frac{ipx}{\hbar}}\, dp$$

$$\left\langle \frac{1}{\sqrt{\;}} e^{\frac{ip'x}{\hbar}} \middle| f(x) \right\rangle =$$

$$\frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-\frac{ip'x}{\hbar}} f(x)\, dx =$$

$$= \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} c(p) \underbrace{\left[\int_{-\infty}^{\infty} e^{i(p-p')x} \, dx \right]}_{2\pi\hbar\, \delta(p-p')} dp$$

$$= c(p')$$

$$\Rightarrow \boxed{c(p') = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-ip'/\hbar x} f(x)\, dx}$$

$c(p)$ is the coefficient function of the expansion of $f(x)$ in a basis of momentum eigenstates.