## Integration Review

1. Calculate the following integrals.
(a) $\int x^{3} \cos (5 x) d x$
(b) $\int x^{2} \sin \left(2 x^{3}\right) d x$
(c) $\int e^{a x} \cos (b x) d x$
(d) $\int_{0}^{2 \pi} \sin (n x) \cos (m x) d x$, where $n, m \in \mathbb{Z}$

## REPRESENTATION OF VECTORS IN VECTOR SPACES

2. Consider the vector space $\mathbb{R}^{2}$, which is the space of all vectors of the form $\hat{\mathbf{v}}=(x, y)$. We know that $\hat{\mathbf{i}}=(1,0)$ and $\hat{\mathbf{j}}=(1,0)$ forms the standard basis for $\mathbb{R}^{2}$. That is, any vector in the space can be created as a linear combination, $\mathbf{v}=c_{1} \hat{\mathbf{i}}+c_{2} \hat{\mathbf{j}}$, of these vectors. However, it is possible to choose other basis vectors and still represent all vectors in the space. Consider defining:

$$
\hat{\mathbf{i}}=\left[\begin{array}{c}
\frac{\sqrt{2}}{2}  \tag{1}\\
\frac{\sqrt{2}}{2}
\end{array}\right], \quad \hat{\mathbf{j}}=\left[\begin{array}{c}
-\frac{\sqrt{2}}{2} \\
\frac{\sqrt{2}}{2}
\end{array}\right]
$$

The previous vectors form another orthonormal-basis for $\mathbb{R}^{2}$.
(a) Show that the vectors are orthonormal by verifying the inner-products $\hat{\mathbf{i}} \cdot \hat{\mathbf{j}}=0$ and $\hat{\mathbf{i}} \cdot \hat{\mathbf{i}}=\hat{\mathbf{j}} \cdot \hat{\mathbf{j}}=1$.
(b) Show that any vector for $\mathbb{R}^{2}$ can be created as a linear combination of $\hat{\mathbf{i}}, \hat{\mathbf{j}}$. That is, given,

$$
\hat{\mathbf{x}}=\left[\begin{array}{l}
x_{1}  \tag{2}\\
x_{2}
\end{array}\right]=c_{1} \hat{\mathbf{i}}+c_{2} \hat{\mathbf{j}}
$$

show that $c_{1}, c_{2}$, can be found in terms of $x_{1}$ and $x_{2}$.
Hint: Recall that the product $\hat{\mathbf{i}} \cdot \hat{\mathbf{j}}$ is called an inner-product or dot-product and is defined to be $\hat{\mathbf{v}}_{1} \cdot \hat{\mathbf{v}}_{2}=$ $\left(x_{1}, y_{1}\right) \cdot\left(x_{2}, y_{2}\right)=x_{1} x_{2}+y_{1} y_{2}$. For part (b) you may want to consider taking inner-products on both sides of the equation.
3. In the previous problem the vector space is $\mathbb{R}^{2}$. However, there are many other mathematical objects that when grouped together satisfy the same rules as the vectors in $\mathbb{R}^{2}$. One such grouping is the set of all $2 \pi$-periodic functions. We call the space of all $2 \pi$-periodic functions an abstract vector space. Each 'vector' in this space is a function, $f$, which has the property that $f(x+2 \pi)=f(x)$. Our goal is to represent any vector in the space by a linear combination of a set of standard basis vectors. Our hope is that there exists an orthonormal basis for this space so that the coefficients in the sum are easy to calculate. Before we do this we must define an inner product. Choose:

$$
\begin{equation*}
f(x) \cdot g(x)=\int_{-\pi}^{\pi} f(x) g(x) d x . \tag{3}
\end{equation*}
$$

Let, $f(x)=\cos (x)$ and $g(x)=\sin (x)$ and show the following orthogonality relations:
(a) $f(n x) \cdot f(m x)=\pi \delta_{n m}$, where $n, m \in \mathbb{Z}$.
(b) $g(n x) \cdot g(m x)=\pi \delta_{n m}$, where $n, m \in \mathbb{Z}$.
(c) $f(n x) \cdot g(m x)=0, \quad n . m \in \mathbb{Z}$, no matter the choice of $n$ and $m$.

Hint: Here the function $\delta_{n m}$ is called the Kronecker delta function and is defined to be zero if $n \neq m$ and one if $n=m$.
Comment: These relationships are important and is a step in showing that the following collection of 'vectors'

$$
\begin{equation*}
\left\{\frac{1}{\sqrt{\pi}}, \frac{\cos (x)}{\sqrt{\pi}}, \frac{\sin (x)}{\sqrt{\pi}}, \frac{\cos (2 x)}{\sqrt{\pi}}, \frac{\sin (2 x)}{\sqrt{\pi}}, \frac{\cos (3 x)}{\sqrt{\pi}}, \frac{\sin (3 x)}{\sqrt{\pi}}, \ldots\right\} \tag{4}
\end{equation*}
$$

forms an orthonormal basis for the space of all $2 \pi$-periodic functions.

## Fourier Series - Introduction

4. Let $f(x)=x^{2},(-\pi<x<\pi)$ be a $2 \pi$-periodic function.
(a) Sketch a graph $f$ on $[-4 \pi, 4 \pi]$.
(b) Go to http://www.tutor-homework.com/grapher.html and graph the following functions,

$$
\begin{aligned}
& f_{1}(x)=\frac{\pi^{2}}{3}-4\left(\cos (x)-\frac{\cos (2 x)}{4}+\frac{\cos (3 x)}{9}\right) \\
& f_{2}(x)=x^{2}
\end{aligned}
$$

(c) Print your results and comment on your three graphs.

## 5. Qualitative Research

(a) Go to http://en.wikipedia.org/wiki/Fourier_series and read the introductory material on Fourier Series and describe in your own words some of the applications of Fourier Series.
(b) Using the Java Applet found at, http://www.sunsite.ubc.ca/LivingMathematics/V001N01/UBCExamples/ Fourier/fourier.html, use truncated Fourier Series to approximate the saw-tooth function. What occurs at the points discontinuity?
(c) Read, as much as you can, of http://en.wikipedia.org/wiki/Gibbs_phenomenon. The sum of a finite, or infinite amount of periodic functions is periodic. Is this always true for both finite and infinite sums of continuous functions? Can you think of a counterexample?

