Reading Today: 12.3

Last time

$$\eta^\mu = \begin{pmatrix}\gamma c\\ \gamma\vec u\end{pmatrix} \quad \text{multiply by } m$$

↑ 4-velocity

$$p^\mu = \begin{pmatrix}\gamma c\\ \gamma m\vec u\end{pmatrix} = \begin{pmatrix}\text{rel. energy}/c\\ \text{rel. momentum}\end{pmatrix}$$

↑ 4-momentum

$$p_\mu p^\mu = -\frac{E^2}{c^2} + p^2 = \text{invariant}$$

Dynamics: $\sum \vec F \neq m\vec a \Rightarrow \sum \vec F = \frac{d\vec p}{dt}$

Find $\frac{d\vec p}{dt}$ in terms of $\underset{\text{vel.}}{\vec u}, \underset{\text{acc.}}{\vec a}$

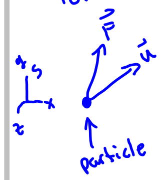
$$\frac{d\vec p}{dt} = \gamma m\vec a + m\gamma^3 \frac{u}{c^2}\underbrace{\frac{du}{dt}}_{\vec a\cdot\hat u}\vec u$$

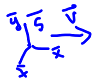
$$= \gamma m\vec a + m\gamma^3 \frac{u}{c^2}\vec a\cdot\hat u\,\vec u$$

$$\gamma = \frac{1}{\sqrt{1-u^2/c^2}} \qquad = \gamma m\left(\vec a + \frac{\gamma^2(\vec a\cdot\vec u)\vec u}{c^2}\right)$$

Can we make a 4-vector out of $\frac{d\vec p}{dt}$?

How does $\frac{d\vec p}{dt}$ transform → that's how forces transform





$$\bar{\vec F}_\perp = \frac{\vec F_\perp}{\gamma(1-\beta u_\parallel/c)}$$

$$\bar F_\parallel = \frac{F_\parallel - \beta(\vec u\cdot\vec F)/c}{1-\beta u_\parallel/c}$$

$\perp \equiv \perp$ to $\vec v$

$\parallel \equiv \parallel$ to $\vec v$

$$\beta = \frac{v}{c}\ ; \ \gamma = \frac{1}{\sqrt{1-v^2/c^2}}$$

$\vec{p}$ has a 4-generalization

The problem comes for $\frac{d}{dt}$

Rather than use t, you can use $\tau \equiv \frac{t}{\gamma}$ (proper time, same in all ref frames).

time in rest frame of the object

$$\gamma = \frac{1}{\sqrt{1-u^2/c^2}}$$

$$\frac{d\vec{p}}{dt} = \frac{d\vec{p}}{d\tau}\frac{d\tau}{dt} = \frac{d\vec{p}}{d\tau}\frac{1}{\gamma} \Rightarrow \frac{d\vec{p}}{d\tau} = \gamma\frac{d\vec{p}}{dt}$$

You can define a new force,

Minkowski Force: $\vec{K} = \gamma \vec{F}$

regular force.

This does have a 4-generalization!

$$K^\mu \equiv \frac{dp^\mu}{d\tau} = \begin{pmatrix} \frac{dp^0}{d\tau} \\ \gamma\vec{F} \end{pmatrix} \qquad p^0 = \frac{E}{c} \quad \rightarrow \frac{dp^0}{d\tau} = \frac{1}{c}\frac{dE}{d\tau}$$

$K_\mu K^\mu = ?$ (only in terms of F, u)

$$K_\mu K^\mu = \frac{F^2 - (\vec{F}\cdot\vec{u})^2/c^2}{1-u^2/c^2} \qquad \frac{dE}{dt} = \vec{F}\cdot\frac{d\vec{l}}{dt}$$

Ok. let's talk $\vec{E}, \vec{B}$. How do they transform going from frame S to a frame $\bar{S}$ that is moving w/ vel. $\vec{v} = v\hat{x}$ w/ respect to S?

$$\bar{E}_x = E_x \qquad \bar{E}_y = \gamma(E_y - vB_z) \qquad \bar{E}_z = \gamma(E_z + vB_y)$$

$$\bar{B}_x = B_x \qquad \bar{B}_y = \gamma\left(B_y + \frac{v}{c^2}E_z\right) \qquad \bar{B}_z = \gamma\left(B_z - \frac{v}{c^2}E_y\right)$$

There is no 4-generalization for $\vec{E}$. or $\vec{B}$. But there is for both together

If I had a matrix in 3D, how did I rotate it?

$$R Z R = \bar{Z}$$

In rel., tensors, you just have to multiply by

$$\bar{t}^{\mu\alpha} = \underbrace{\frac{\partial \bar{x}^{\mu}}{\partial x^{\nu}}}_{\Lambda_{\nu}^{\mu}} \underbrace{\frac{\partial \bar{x}^{\alpha}}{\partial x^{\beta}}}_{\Lambda_{\beta}^{\alpha}} t^{\nu\beta}$$

If we define

$$F^{\mu\nu} \equiv \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix}$$

← That's contravariant and it transforms like on the last page.

$$G^{\mu\nu} \equiv \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z/c & E_y/c \\ -B_y & E_z/c & 0 & -E_x/c \\ -B_z & -E_y/c & E_x/c & 0 \end{pmatrix}$$

$$G^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}$$

$$\epsilon = \begin{cases} 1, & 0,1,2,3 \text{ even perm.} \\ -1, & \text{odd perm} \\ 0, & \text{if any are same.} \end{cases}$$

As we're generalizing, how about charge?

From cons. of charge $\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = \emptyset$

$$\frac{\partial (c\rho)}{\partial (ct)} + \vec{\nabla}\cdot\vec{J} = \emptyset$$

$$\frac{\partial}{\partial x^0}\hat{x}^0 + \frac{\partial}{\partial x^1}\hat{x}^1 + \cdots$$

If I define $J^{\mu} = \begin{pmatrix} c\rho \\ \vec{J} \end{pmatrix}$

this eqn looks like $\partial_{\mu} J^{\mu} = \emptyset$

Maxwell's eqns can be written as

$\partial_\nu F^{\mu\nu} = \mu_0 J^\mu$ $\quad\quad$ $\partial_\nu G^{\mu\nu} = 0$

You can rewrite this in terms of $F^{\mu\nu}$

$\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0$

for any α, β, γ (0, 1, 2, 3)

Potentials:

$$A^\mu = \left(\frac{V}{c}, A_x, A_y, A_z\right)$$

Relationship between the fields and potentials can be written as

$$F^{\mu\nu} = \frac{\partial A^\nu}{\partial x_\mu} - \frac{\partial A^\mu}{\partial x_\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

We know $\partial_\nu F^{\mu\nu} = \mu_0 J^\mu$

$$\partial_\nu \partial^\mu A^\nu - \partial_\nu \partial^\nu A^\mu = \mu_0 J^\mu$$

$$\partial^\mu \underbrace{\partial_\nu A^\nu}$$

$\frac{1}{c^2}\frac{\partial}{\partial t} V + \vec{\nabla}\cdot\vec{A} = 0$ in Lorentz gauge.

$$\boxed{\partial_\nu \partial^\nu A^\mu = \mu_0 J^\mu}$$

with $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$

and $K^\mu = F^{\mu\nu} J_\nu$

All of E & M is here

EM Energy / Momentum

stress-energy tensor

$$T^{\mu\nu} \equiv \frac{1}{\mu_0}\left(F^{\mu\alpha} F_{\alpha}{}^{\beta} - \frac{1}{4} g^{\mu\nu} F_{\delta\gamma} F^{\delta\gamma} \right)$$

$$\begin{pmatrix} \propto u_{em} & \propto \vec{S} \\ \propto \vec{S} & \propto \overleftrightarrow{T} \end{pmatrix}$$

cons. of energy/momentum

$$\boxed{\partial_\mu T^{\mu\nu} = \frac{1}{c} J_\beta F^{\beta\nu}}$$

Prof : 589

Course : 0462

Dep : 15

Sec : A