1-D waves: $\frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$

3-D waves! $\nabla^2 f = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \vec{\nabla}\cdot\vec{\nabla}$$
{div of grad}

Our general solution becomes

$$\tilde{f} = \tilde{A}\, e^{i(\vec{k}\cdot\vec{r} - \omega t)} = \tilde{A} e^{i(k_x x + k_y y + k_z z - \omega t)}$$

$$v_{wave} = \frac{\omega}{k} \; ; \; k = |\vec{k}| = \sqrt{k_x^2 + k_y^2 + k_z^2}$$

$\boxed{\text{Show } \tilde{f} \text{ satisfies the 3-D wave eq}^n.}$

Any function in 3-D can be written as

$$\tilde{f}(\vec{r},t) = \int_{\text{all } \vec{k}} \tilde{A}(\vec{k})\, e^{i(\vec{k}\cdot\vec{r} - \omega t)}\, d^3k$$

(or) $$\tilde{f}(x,y,z,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{A}(k_x,k_y,k_z)\, e^{i(k_x x + k_y y + k_z z - \omega t)} \cdot dk_x\, dk_y\, dk_z$$

$\boxed{\text{Show that } \nabla^2\vec{E} = \mu_0\epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} \; ; \; \nabla^2\vec{B} = \mu_0\epsilon_0\frac{\partial^2\vec{B}}{\partial t^2} \text{ in absence of } \rho, \vec{J} \text{ (free space)}}$

hint: Take $\nabla\times$(Max 3) and $\nabla\times$(Max 4)

This shows the speed of light $c = \frac{1}{\sqrt{\mu_0\epsilon_0}}$

In general, we can write.

$$\vec{E} = \vec{E}_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)} \qquad \vec{B} = \vec{B}_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$E_x = E_{0x} e^{i(\vec{k}\cdot\vec{r} - \omega t)} \qquad ''$$

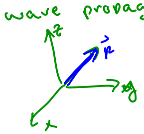
$$E_y = E_{0y} e^{i(\vec{k}\cdot\vec{r} - \omega t)} \qquad ''$$

$$E_z = E_{0z} e^{i(\vec{k}\cdot\vec{r} - \omega t)} \qquad ''$$

This isn't the whole story.

Let's say we have a wave propagating in the $\vec{k}$ direction

Instead, let's pick the z-direction to line up with $\vec{k}$.



$$\Rightarrow \vec{k} = k_z \hat{z}$$

$$\vec{E} = \vec{E}_0 e^{i(k_z z - \omega t)} \qquad \vec{B} = \vec{B}_0 e^{i(k_z z - \omega t)}$$

$$\vec{\nabla}\cdot\vec{E} = \cancel{\frac{\partial E_x}{\partial x}}^{\,\emptyset} + \cancel{\frac{\partial E_y}{\partial y}}^{\,\emptyset} + \frac{\partial E_z}{\partial z} = i k_z E_{0z} e^{i(k_z z - \omega t)} = \emptyset$$

because $\rho = \emptyset$

$$\Rightarrow E_{0z} = \emptyset \Rightarrow \boxed{\vec{E}_k = 0} \Rightarrow$$ $\vec{E}$ waves are transverse.

Same follows for $\vec{B}$.

$$\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} = +i\omega \vec{B} = i\omega\left[B_{0x}\hat{x} + B_{0y}\hat{y}\right]$$

$$\vec{\nabla}\times\vec{E} = \left(\cancel{\frac{\partial E_z}{\partial y}}^{\,\emptyset} - \frac{\partial E_y}{\partial z}\right)\hat{x} + \left(\frac{\partial E_x}{\partial z} - \cancel{\frac{\partial E_z}{\partial x}}^{\,\emptyset}\right)\hat{y} + \emptyset.$$

$$= -\frac{\partial E_y}{\partial z}\hat{x} + \frac{\partial E_x}{\partial z}\hat{y}$$

$$= -i k_z E_y \hat{x} + i k_z E_x \hat{y}$$

$$\Rightarrow \vec{B} = \frac{1}{\omega}\left[-k_z E_y \hat{x} + k_z E_x \hat{y}\right]$$

$$\vec{k}\times\vec{E} = k_z E_x \hat{y} - k_z E_y \hat{x}$$

$$\vec{B} = \frac{\vec{k}\times\vec{E}}{\omega} = \frac{1}{c}(\hat{k}\times\vec{E})$$

$\Rightarrow$ The real general soln for a plane EM wave is

$$\vec{E} = \left[\tilde{E}_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)}\right]\hat{n}$$

$\hat{n}$ = polarization ; $\hat{n}\cdot\hat{k}$ must be $\emptyset$

$$\vec{B} = \frac{1}{\omega}\vec{k}\times\vec{E} = \frac{1}{c}(\hat{k}\times\vec{E})$$

Let's talk about time averages.

$$\langle E^2 \rangle = \vec{E}\cdot\vec{E}^* = \tilde{E}_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)} \cdot \tilde{E}_0^* e^{-i(\vec{k}\cdot\vec{r} - \omega t)}$$

$$= \tilde{E}_0 \tilde{E}_0^* = E_0 e^{i\delta} E_0 e^{-i\delta} = E_0^2$$

$$\langle \vec{S} \rangle = \langle \frac{1}{\mu_0}\vec{E}\times\vec{B} \rangle = \frac{1}{\mu_0}\vec{E}\times\vec{B}^*$$

More generally for non-plane waves

$$\langle \vec{S} \rangle = \mathrm{Re}\left[\frac{1}{\mu_0}\vec{E}\times\vec{B}^*\right], \quad \langle u_{em} \rangle = \frac{\epsilon_0}{2}\vec{E}\cdot\vec{E}^* + \frac{1}{2\mu_0}\vec{B}\cdot\vec{B}^*$$

Find $\langle u_{em} \rangle$ an $\langle \vec{S} \rangle$ for a plane wave

(a) in term of just E_0

(b) " " " " B_0

(c) what's the ratio of the energy in $\vec{E}$
" " " $\vec{B}$