E. Kreyszig, Advanced Engineering Mathematics, 9^{th} ed. Section 11.7, pgs. 506-513

Lecture: Fourier Integral Module: 11

Suggested Problem Set: {3, 7, 9, 14, 15} March 30, 2009

E. Kreyszig, Advanced Engineering Mathematics, 9^{th} ed. Section 11.8, pgs. 513-517

Lecture: Fourier Sine/Cosine Transform Module: 11

Suggested Problem Set: {1, 5, 6} March 30, 2009

E. Kreyszig, Advanced Engineering Mathematics, 9^{th} ed. Section 11.9, pgs. 518-528

Lecture: Complex Fourier Transform Module: 11

Suggested Problem Set: {2, 3, 9, 14(a)} March 30, 2009

Quote of Lecture 11

Everybody's talking 'bout the stormy weather and what's a man do to but work out whether it's true? Looking for a man with a focus and a temper who can open up a map and see between one and two. Time to get it before you let it get to you. Here he comes now; stick to your guns and let him through.

Sonic Youth : Teenage Riot (1988)

1. Overview

Now that we have ended the study of Fourier series, which are used to represent piecewise continuous functions that have the additional feature of being periodic, we as the question:

- Can these Fourier methods be applied to functions that do not have the extra structure of periodicity?

Why would we like to do this? Well, we have seen already that Fourier series allow us to gather from a function its particular frequencies of oscillation and also their associated amplitudes. With this description one can then gain insight into the overall 'energy' of the function as well as how each of the oscillatory modes contributes to this energy. Without the Fourier representation much of this information is inaccessible. If we can port these methods over to functions without a periodic substructure then we may find similar interpretations and consequently Fourier methods would be invaluable to an extremely large class of physically relevant functions.

So, how should we go about this? Well, we must first notice that one of the fundamental assumption on functions with a Fourier series representation is that they can be defined by using a finite portion of the real-line. This finite portion is considered to be the principle-period and from this information the rest of the function is constructed by repeating the functions graph on this domain to the rest of $\mathbb{R}$. If we consider this principle-period to be the $(-L, L)$, which is a finite portion of $\mathbb{R}$, then we can destroy these concepts by taking the limit $L \to \infty$. Letting L become unbounded gives rise to a plausible heuristic derivation, which results in the well-celebrated complex Fourier transform, or just Fourier transform for short. Though the text chooses to consider first the limit, $L \to \infty$, to arrive at the Fourier integral and symmetrically exploited to define the Fourier cosine/sine transforms, which upon reconsideration is used to define the more general complex Fourier transform, we instead use the Fourier integral to go directly to the complex Fourier transform and show that the intermediary results as special cases.[1] In the following points I outline the key points of logic used in this derivation:

(1) Consider the Fourier series representation of a $2L$-periodic function in its full form:

$$f(x) = \frac{1}{2L}\int_{-L}^{L} f(v)dv + \sum_{n=1}^{\infty}\left[\frac{1}{L}\int_{-L}^{L} f(v)\cos(\omega_n v)dv\right]\cos(\omega_n x) + \left[\frac{1}{L}\int_{-L}^{L} f(v)\sin(\omega_n v)dv\right]\sin(\omega_n x), \quad \omega_n = \frac{n\pi}{L}.$$

(2) Assume that the function f is absolutely integrable,

$$\int_{-\infty}^{\infty} |f(x)|dx < \infty,$$

and 'take the limit,' $L \to \infty$. [2] Once we have 'taken' this limit we arrive at the Fourier integral representation of a function that need not be periodic, [3]

$$f(x) = \int_{0}^{\infty} (A(\omega)\cos(\omega x) + B(\omega)\sin(\omega x))\, d\omega,$$

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos(\omega x)dx, \quad B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin(\omega x)dx$$

(3) Lastly, consider the Fourier integral representation of the function f in its full-form,

$$f(x) = \int_{0}^{\infty}\left(\left[\frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos(\omega v)dv\right]\cos(\omega x) + \left[\frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin(\omega v)dv\right]\sin(\omega x)\right)d\omega,$$

and again after some decent algebra we arrive at the complex Fourier transform pair,

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx. \quad f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega x}d\omega,$$

Before we begin using this integral transform we make a note of it similarity to the complex Fourier series,

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx. \qquad f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega x}d\omega, \quad \omega \in \mathbb{R},$$

$$c(\omega_n) = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-i\omega_n x}dx, \qquad f(x) = \sum_{n=-\infty}^{\infty} c(\omega_n)e^{i\omega_n x}, \quad \omega_n = \frac{n\pi}{L},$$

and highlight connections with previous logic stressing that the connection is found in the periodicity destroying limit where $L \to \infty$: [4]

- Roll of coefficients - In both cases the 'forward-integral' converts the function f into amplitude information $\hat{f}$ in the case of Fourier transform and $c(\omega_n)$ in the case of Fourier series. These coefficients are then used represent f as a linear combination of oscillatory functions $e^{i\omega x}$ in the case of transform and $e^{i\omega_n x}$ in the case of series. The major difference between the two methods is that if the function f is periodic then frequency information is needed only for integer multiples of π/L while in the case that f is not periodic increased frequency information is needed.
- Energy - As with Fourier series the integral-conversion of the function f to amplitude information makes accessible 'energy' information associated with the function f.

$$E_{series} \propto \sum_{-\infty}^{\infty} |c(\omega_n)|^2, \quad E_{transform} \propto \int_{-\infty}^{\infty} |\hat{f}(\omega)|^2 d\omega$$

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[1] The book does this to mimic the logic used for building the Fourier series. This correspondence is nice but drags one and one-half lectures of heavy symbolic manipulation into three. I favor the shorter derivation since it is unlikely that you will need to ever repeat them.

[2] Much is hidden here. Formally, we have to interchange two limiting processes. This brings a good amount of fear to the table and mathematically we must dispel this fear by convergence tests. It is unfortunate that the required uniform convergence cannot always be guaranteed and that because of this increased mathematical machinery must be constructed and applied. The short story is that the methods work quite well and shouldn't keep you up at night.

[3] We have similar interpretations of this as we did with Fourier series. Namely, we represent the function f as a linear combination of trigonometric functions of varying frequency. The weights of this linear combination are given by integrating the function f and are again thought of as amplitudes information for each oscillatory mode. The major difference we see here is that since the function is more complicated, in that it does not have the additional structure of periodicity, we are required to use every possible frequency of oscillation for the trigonometric functions. This results in the use of a continuous infinite sum as opposed to a discretely infinite sum in the case of Fourier series.

[4] One can recover the Fourier series as a special case of the Fourier transform by using the so-called Dirac delta 'function,' but this is a cheat since it really isn't a function at all and exists so that we can make integrals perform tricks we desire.

- Symmetry Arguments - Lastly, we have similar symmetry arguments. With Fourier series we have,[5]

$$f(-x) = f(x) \implies f(x) = \sum_{-\infty}^{\infty} c(\omega_n)e^{i\omega_n x} = a(0) + \sum_{n=1}^{\infty} a(\omega_n)\cos(\omega x)$$
$$f(-x) = -f(x) \implies f(x) = \sum_{-\infty}^{\infty} c(\omega_n)e^{i\omega_n x} = \sum_{n=1}^{\infty} b(\omega_n)\sin(\omega_n x).$$

While for Fourier transforms we have the following statements,

$$f(-x) = f(x) \iff \hat{f}(-\omega) = \hat{f}(\omega) \implies f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega x}d\omega = \sqrt{\frac{2}{\pi}}\int_{0}^{\infty} \hat{f}(\omega)\cos(\omega x)d\omega$$
$$f(-x) = -f(x) \iff \hat{f}(-\omega) = -\hat{f}(\omega) \implies f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega x}d\omega = \sqrt{\frac{2}{\pi}}\int_{0}^{\infty} \hat{f}(\omega)\sin(\omega x)d\omega.$$

The take-home message is that if a function has even(odd) symmetry then the complex Fourier series reduces to a Fourier cosine(sine) series while the complex Fourier transform reduces to a Fourier cosine(sine) transform. [6]

2. Lecture Goals

Our goals with this material will be:

- Construct from the Fourier series an integral representation for functions, which may not have a periodic structure.
- Understand the analogies between the Fourier series and Fourier integral representation of a function.

3. Lecture Objectives

The objectives of these lessons will be:

- Consider the consequences of the limit, $L \to \infty$, on the Fourier series as a means to 'derive' the Fourier integral.
- From the Fourier integral derive the Fourier transform.
- Compare and contrast the symmetry and spectral decomposition properties of Fourier transform to complex Fourier series.

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[5] There is a fair amount of algebra needed to show these statements. However, if we trust that complex Fourier series are equivalent to real Fourier series, which we should, then these equalities MUST be true.

[6] Note that these statements define the inverse transforms (taking amplitude data and representing f as a linear combination of oscillatory modes). Symmetry arguments are needed to simplify the forward transforms.