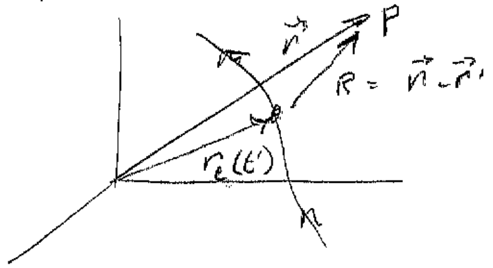
Lienard Wiechert Potentials

potentials due to a moving point charge.



r_e

describe the point charge as a δ-function

$$\Phi(\vec{r}, t) = \int \frac{\rho(\vec{r}', t - |\vec{r}-\vec{r}'|/c)}{|\vec{r}-\vec{r}'|} d^3r'$$

$$= \iint \frac{\rho(\vec{r}', t')}{|\vec{r}-\vec{r}'|} \delta\left(t' - (t - |\vec{r}-\vec{r}'|/c)\right) d^3r'\, dt'$$

charge is a point at $\vec{r}' = \vec{r}_e(t')$

$$\rho \to q\,\delta(\vec{r}' - \vec{r}_e(t'))$$

and integrate over $\vec{r}'$

$$\Phi = \int q \frac{\delta\left(t' - (t - |\vec{r}-\vec{r}_e|/c)\right)}{|\vec{r}-\vec{r}_e|} dt'$$

remember $\vec{r}_e(t')$ depends on t'

change variables:

$$t'' = t' - t + |\vec{r}-\vec{r}_e|/c$$

$$dt'' = dt'\left(1 + \frac{1}{c}\frac{d}{dt'}|\vec{r}-\vec{r}_e(t')|\right) \qquad \text{let } \vec{R} = \vec{r}-\vec{r}_e$$

$$= dt'\left(1 + \frac{1}{c}\left(\frac{\partial}{\partial x_e}|\vec{R}|\frac{d\vec{x}_e}{dt'} + \frac{\partial}{\partial y}|\vec{R}|\frac{d}{dt'}\vec{y}_e + \quad\right)\right)$$

$$= dt'\left(1 + \frac{1}{c}\left(\nabla_{r_e}|\vec{R}| \cdot \frac{d}{dt'}\vec{r}_e\right)\right)$$

where $\nabla_{r_e} \vec{R} = \nabla_{r_e} |\vec{r} - \vec{r}_e| = -\frac{\vec{R}}{R}$

by working out derivatives of $\sqrt{\sum (x_i - x_{ei})^2}$

and $\frac{d\vec{r}_e}{dt'} = \vec{u}$ velocity of particle.

define $\vec{\beta} = \vec{u}/c$

now put terms together:

$$dt'' = dt' \left(1 - \vec{\beta}\cdot\vec{R}/R\right)$$

or

$$dt' = dt'' \frac{R}{R - \vec{\beta}\cdot\vec{R}}$$

go back to integral for Φ

$$\Phi(\vec{r},t) = q \int \frac{\delta(t'')}{\cancel{R(t')}} \frac{\cancel{R(t')}\ dt''}{R(t') - \vec{\beta}(t')\cdot\vec{R}(t')}$$

$$= q \left.\frac{1}{R(t') - \vec{\beta}(t')\cdot\vec{R}(t')}\right|_{t''=0}$$

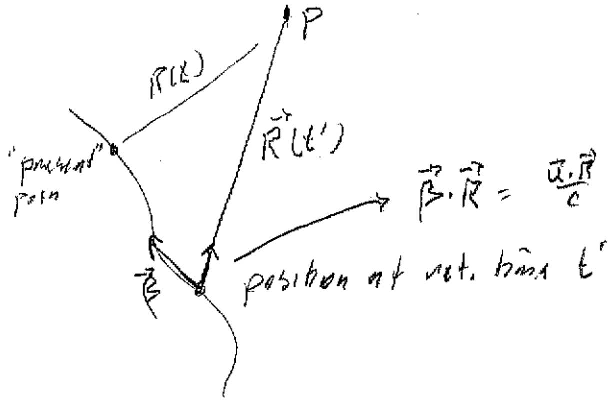
$$t'' = t' - t + R(t')/c$$

so evaluating at $t''=0$ is same as evaluating at the retarded time.

$$\rightarrow \Phi(r,t) = \frac{q}{[R - \vec{\beta}\cdot\vec{R}]} = \frac{q}{[KR]} \qquad K = 1 - \frac{\vec{\beta}\cdot\vec{R}}{R}$$

$$\vec{A}(\vec{r},t) = \frac{q[\vec{\beta}]}{[R - \vec{\beta}\cdot\vec{R}]} = q\frac{\vec{\beta}}{[KR]}$$

discussion



$\vec{\beta}\cdot\vec{R} = \frac{\vec{u}\cdot\vec{R}}{c}$ subtract this component

Φ is increased slightly if particle is moving toward observer at P

$$\sim \quad \frac{q}{R - \vec{\beta}\cdot\vec{R}}$$

R

Lienard - Wiechert Fields

two methods:

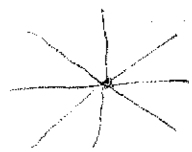
- differentiate L-W potentials - Griffiths
- take limit of field equations for ρ, J - 4m

$$\rightarrow \vec{E} = e\left[\frac{(\hat{R}-\vec{\beta})(1-\beta^2)}{K^3R^2} + \frac{\hat{R}\times((\hat{R}-\vec{\beta})\times\vec{a})}{c^2K^3R}\right]_{tr}$$

$$\vec{B} = [\hat{R}] \times \vec{E}$$

$$\vec{E} = \text{veloc. term} + \text{accel. term.}$$

<u>veloc.</u> term: as $\vec{\beta} = \vec{v}/c \rightarrow 0 \qquad \vec{E} = \dfrac{e\hat{R}}{R^2}$



$\vec{E}$ components $\parallel$ to both $\hat{R}$ and $\vec{\beta}$

$\vec{B} \perp$ to $\vec{E}$ and $\hat{R}$

both $\propto 1/R^2$

<u>accel</u> term: $\propto a$ but $\perp$ to $\hat{a}$

$E, B \propto 1/R$

Calculation of $\vec{S} = \frac{c}{4\pi} \vec{E}\times\vec{B}$ will have 3 types of terms a, v

$$S_{v,v} \sim E_v, B_v \propto 1/[R]^4$$

$$S_{av} \text{ and } S_{va} \propto 1/[R]^3$$

$$S_{aa} \propto 1/[R]^2$$

far enough away: only S_{aa} = radiation

Example: radiation damping

electron bound in SHO potential. $F = \alpha x$ Hooke's law

$t=0$ stretch, release.

$m_e a(t) = \alpha x(t)$

$\ddot{x} = \frac{\alpha}{m_e} x \rightarrow x = x_0 \cos(\omega_0 t) \qquad \omega_0 = \sqrt{\alpha / m_e}$

$\therefore$ no damping, $a(t) = \frac{\alpha}{m_e} x_0 \cos(\omega_0 t) = \omega_0^2 x_0 \cos(\omega_0 t)$

acceleration $\rightarrow$ radiation, energy loss.

$$P = \frac{2}{3}\frac{e^2}{c^3} a^2(t) = \frac{2}{3}\frac{e^2}{c^3}\omega_0^4 x_0^2 \cos^2(\omega_0 t)$$

more power emitted at turning points

time avg: (cycle avg) $\langle \cos^2 \omega_0 t \rangle = \frac{1}{2}$

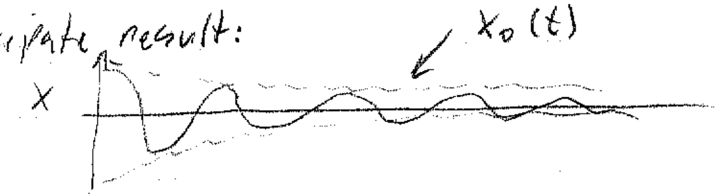
$$\langle P \rangle = \frac{1}{3}\frac{(e x_0)^2 \omega_0^4}{c^3}$$

$e x_0 = p$ = dipole moment. $\quad \langle P \rangle = \frac{p^2 \omega_0^4}{3c^3}$

if electron is driven (say by EM wave) $\rightarrow$ Rayleigh scattering.

for our problem, electron gradually loses energy

anticipate result:



$x(t) = x_0(t) \cos(\omega_0 t)$

$\hookrightarrow$ envelope.

This is a tough, nonlinear problem:

$$-P = \frac{d}{dt}(E_{tot}) \Rightarrow \frac{d}{dt}\left(\frac{1}{2} m \dot{x}^2 + \frac{1}{2} m \omega_0^2 x^2\right) = -\frac{2}{3}\frac{e^2 (\ddot{x})^2}{c^3}$$

w/o damping, $\mathcal{E}_{tot} = \frac{1}{2} m \omega_0^2 x_0^2 \sin^2 \omega_0 t$
$$+ \frac{1}{2} m \omega_0^2 x_0^2 \cos^2 \omega_0 t$$
$$= \frac{1}{2} m \omega_0^2 x_0^2 \quad = \text{constant.}$$

$\therefore$ slow decrease in energy $\rightarrow$ slow decrease in ampl. x_0

write $$\frac{d}{dt}\left(\frac{1}{2} m \omega_0^2 x_0^2(t)\right) = -\langle P \rangle = $$
$$= -\frac{1}{3} \frac{e^2 \omega_0^4}{c^3} x_0^2(t)$$

or write as diff eq. for $\mathcal{E}_{tot}$: $\quad x_0^2 = \frac{2\mathcal{E}_{tot}}{m\omega_0^2}$

$$\frac{d}{dt}\mathcal{E}_{tot} = -\frac{1}{3} \frac{e^2 \omega_0^4}{c^3} \cdot \frac{2\mathcal{E}_{tot}}{m\omega_0^2}$$
$$= -\frac{2}{3} \frac{e^2}{mc^2} \frac{\omega_0^2}{c} \mathcal{E}_{tot}$$

$\frac{e^2}{mc^2} = r_e$ classical electron radius $\sim 3 \times 10^{-15}$ m

$$\dot{\mathcal{E}}_{tot} = -\gamma_R \mathcal{E}_{tot} \qquad \gamma_R = \frac{2}{3} r_e \omega_0^2 / c$$

$$\mathcal{E}_{tot} = \mathcal{E}_0 e^{-\gamma_R t}$$ radiation damping.

$$\gamma_R = \frac{2}{3} r_e \frac{4\pi^2 c}{\lambda^2}$$

At $\lambda = 500$ nm $\quad 1/\gamma_R = 10$ ns $\approx$ natural decay time in atoms.

Can think of damping as force on electron by field.
$\rightarrow$ "radiation reaction" very tough problem theoretically.
in antennas $\rightarrow$ radiation resistance.