MATH348-Advanced Engineering Mathematics		**Homework: PDEs - Part II**
Conservation Laws, Source Terms, Diffusion in $\mathbb{R}^{2+1}$, Chain Rule, SL-Problems		
Text: 12.8,12.9	Lecture Notes : N/A	Lecture Slides: N/A

Quote of Homework Five
And the feeling is that there's something wrong, 'cause I can't find the words, and I can't find the songs.
Radiohead : Stop Whispering (1993)

1. Conservation Laws in One-Dimension

Recall that the conservation law encountered during the derivation of the heat equation was given by,

$$\frac{\partial u}{\partial t} = -\kappa \nabla \cdot \boldsymbol{\phi} = -\kappa \operatorname{div}(\boldsymbol{\phi}), \tag{1}$$

which reduces to

$$\frac{\partial u}{\partial t} = -\kappa \frac{\partial \phi}{\partial x},\ \kappa \in \mathbb{R} \tag{2}$$

in one-dimension of space.[1] In general, if the function $u = u(x,t)$ represents the density of a physical quantity then the function $\phi = \phi(x,t)$ represents its flux. If we assume the ϕ is proportional to the negative gradient of u then, from (2), we get the one-dimensional heat/diffusion equation. [2]

1.1. **Transport Equation.** Assume that ϕ is proportional to u to derive, from (2), the convection/transport equation, $u_t + cu_x = 0 \ \ c \in \mathbb{R}$.

1.2. **General Solution to the Transport Equation.** Show that $u(x,t) = f(x - ct)$ is a solution to this PDE.

1.3. **Diffusion-Transport Equation.** If both diffusion and convection are present in the physical system then the flux is given by, $\phi(x,t) = cu - du_x$, where $c, d \in \mathbb{R}^+$. Derive from, (2), the convection-diffusion equation $u_t + \alpha u_x - \beta u_{xx} = 0$ for some $\alpha, \beta \in \mathbb{R}$.

1.4. **Convection-Diffusion-Decay.** If there is also energy/particle loss proportional to the amount present then we introduce to the convection-diffusion equation the term λu to get the convection-diffusion-decay equation.[3]

1.5. **General Importance of Heat/Diffusion Problems.** Given that,

$$u_t = Du_{xx} - cu_x - \lambda u. \tag{3}$$

Show that by assuming, $u(x,t) = w(x,t)e^{\alpha x - \beta t}$, (3) can be transformed into a heat equation on the new variable w where $\alpha = c/(2D)$ and $\beta = \lambda + c^2/(4D)$.[4]

2. One Dimensional Heat Equation with Source Term

Given,

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} + F(x,t), \tag{4}$$

where $x \in (0, L)$ and $t \in (0, \infty)$, subject to

$$u_x(0,t) = 0,\ u_x(L,t) = 0, \tag{5}$$

and

$$u(x,0) = g(x). \tag{6}$$

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[1] When discussing heat transfer this is known as Fourier's Law of Cooling. In problems of steady-state linear diffusion this would be called Fick's First Law. In discussing electricity u could be charge density and q would be its flux.

[2] AKA Fick's Second Law associated with linear non-steady-state diffusion.

[3] The u_{xx} term models diffusion of energy/particles while u_x models convection, u models energy/particle loss/decay. The final term should not be surprising! Wasn't the appropriate model for radioactive/exponential decay $Y' = -\alpha^2 Y$?

[4] This shows that the general PDE (3), which models can be solved using heat equation techniques.

2.1. **Cosine Half-Range Expansion.** Let $F(x,t) = e^{-t}\sin\left(\frac{2\pi}{L}x\right)$ be the heat generation function. Find the Fourier cosine half-range expansion of F.

2.2. **General Solution.** Using the previous result, solve for $G_n(t)$ for $n = 0, 1, 2, 3, \ldots$ assuming that $u(x,t) = G_0(t) + \sum_{n=1}^{\infty} \cos\left(\frac{n\pi}{L}x\right) G_n(t)$.

2.3. **Fourier Coefficients.** Assuming that $g(x) = \begin{cases} \frac{2k}{L}x, & 0 < x < \frac{L}{2}, \\ \frac{2k}{L}(L-x), & \frac{L}{2} < x < L \end{cases}$, solve for any unknown constants associated with the general solution.

3. Time Dependent Boundary Conditions

It makes sense to consider time-dependent interface conditions. That is, (4) and (6) subject to

$$u(0,t) = g(t), \quad u(L,t) = h(t), \quad t \in (0,\infty) \tag{7}$$

Show that this PDE transforms into:

$$\frac{\partial w}{\partial t} = c^2 \frac{\partial^2 w}{\partial x^2} - S_t(x,t) \quad , \tag{8}$$

$$x \in (0,L)\,, \qquad t \in (0,\infty)\,, \qquad c^2 = \frac{\kappa}{\rho\sigma}. \tag{9}$$

with boundary conditions and initial conditions,

$$w(0,t) = w(L,t) = 0, \tag{10}$$

$$w(x,0) = F(x), \tag{11}$$

where $F(x) = f(x) - S(x,0)$ and $S(x,t) = \frac{h(t)-g(t)}{L}x + g(t)$. [5]

4. Heat Equation on a bounded domain of $\mathbb{R}^{2+1}$

Suppose that heat is allowed to flow in an $x,y-$plane, of finite area, $A = L_x L_y$, that has been insulated in the $z-$direction and its perimeter.

4.1. **Separation of Variables.** Find three ODEs consistent with the heat equation modeling the physical situation described above.

4.2. **Boundary Value Problems.** Write down the boundary conditions implied by the physical situation above and solve all ODEs, with their corresponding boundary conditions, given by the separation of variables above.

4.3. **Fourier Synthesis.** Apply superposition to the solutions of the ODE/BVPs from the previous step to find the general solution to the heat equation. From the general solution, show that the long-time behavior is to average the initial condition over the plane.

5. Fourier Transforms

5.1. **Dirac-Delta.** $\mathfrak{F}\{f\}$ where $f(x) = \delta(x - x_0)$, $x_0 \in \mathbb{R}$. [6]

5.2. **Decaying Exponential Function.** $\mathfrak{F}\{f\}$ where $f(x) = e^{-k_0|x|}$, $k_0 \in \mathbb{R}^+$.

5.3. **Even Combination of Dirac-Deltas.** $\mathfrak{F}^{-1}\left\{\hat{f}\right\}$ where $\hat{f}(\omega) = \frac{1}{2}\left(\delta(\omega+\omega_0) + \delta(\omega-\omega_0)\right)$, $\omega_0 \in \mathbb{R}$.

5.4. **Odd Combination of Dirac-Deltas.** $\mathfrak{F}^{-1}\left\{\hat{f}\right\}$ where $\hat{f}(\omega) = \frac{1}{2i}\left(\delta(\omega+\omega_0) - \delta(\omega-\omega_0)\right)$, $\omega_0 \in \mathbb{R}$.

5.5. **Horizontal Shifts.** Find $\hat{f}(\omega)$ where $f(x+c)$, $c \in \mathbb{R}$.

5.6. **Triangle Functions.** Find $\mathfrak{F}\{f\}$ where $f(x) = \begin{cases} 1-|x|, & -1 < x < 1, \\ 0, & \text{otherwise} \end{cases}$

5.7. **Hyperbolic Secant Function.** Find $\mathfrak{F}\{f\}$ where $f(x) = \operatorname{sech}(\pi x)$

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[5] A similar transformation can be found for the wave equation with inhomogeneous boundary conditions. The moral here is that time-dependent boundary conditions can be transformed into externally driven (AKA Forced or inhomogeneous) PDE with standard boundary conditions.

[6] Here the δ is the so-called Dirac, or continuous, delta function. This isn't a function in the true sense of the term but instead what is called a generalized function. The details are unimportant and in this case we care only that this Dirac-delta *function* has the property $\int_{-\infty}^{\infty} \delta(x-x_0)f(x)dx = f(x_0)$. For more information on this matter consider `http://en.wikipedia.org/wiki/Dirac_delta_function`. To drive home that this *function* is strange, let me spoil the punch-line. The sampling function $f(x) = \operatorname{sinc}(ax)$ can be used as a definition for the Delta *function* as $a \to 0$. So can a bell-curve probability distribution. Yikes!