PHGN 462 Homework 3

1) (From Pollack and Stump 11.13)

Consider the electromagnetic field

$$\vec{E}(x,y,t) = E_0 \cos\left(\frac{\pi x}{L}\right)\cos\left(\frac{\pi y}{L}\right)\sin(\omega t)\,\hat{k}$$

$$\vec{B}(x,y,t) = B_0\left[-\cos\left(\frac{\pi x}{L}\right)\sin\left(\frac{\pi y}{L}\right)\hat{\imath} + \sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{\pi y}{L}\right)\hat{\jmath}\right]\cos(\omega t)$$

This is an example of a standing wave. Standing waves in cavities show up in a lot of practical places; not the least of which is a laser cavity. This one in particular might be a wave in a long rectangular box stretching out in the z direction, with walls of length L in the x and y directions.

a) Show that this field satisfies the Maxwell equations in vacuum if $\omega = \frac{\sqrt{2}\pi c}{L}$ and $B_0 = \frac{E_0}{\sqrt{2}c}$. Notice in particular that our old familiar $E = cB$ doesn't apply here – that result was derived for plane waves, and this isn't one.

b) Sketch the E and B fields. These are moderately complicated fields and are not trivial to draw. I'd like you to give particular thought to how best to visually represent those fields in this situation. There is no one accepted standard.

2) There's a general method for describing the polarization of electromagnetic waves using what are called Jones vectors. We'll restrict ourselves to describing the electric field in a wave, since once you know E and the direction in which the wave is traveling, you can easily find the orientation and amplitude of B (at least, I hope you can).

Let's say we have an E-field propagating along the z-axis with the form

$$\vec{E} = E_x e^{i(kz-wt-\varphi_x)}\hat{\imath} + E_y e^{i(kz-wt-\varphi_y)}\hat{\jmath}$$

Such a form is totally general, and allows for the possibility that the $\hat{\imath}$ and $\hat{\jmath}$ components of the field are of different amplitudes and also of different phases with respect to one another. We can factor out the part common to both components and write $\vec{E}$ in vector form as follows:

$$\vec{E} = e^{i(kz-wt)}\begin{pmatrix} E_x e^{i\varphi_x} \\ E_y e^{i\varphi_y} \end{pmatrix}$$

The term in parentheses has all the information that's unique to a particular field, showing both the component amplitudes and phases. This term is the Jones vector.

The Jones vectors for horizontally and vertically polarized light with unit amplitude are $\overrightarrow{E_x} = \binom{1}{0}$ and $\overrightarrow{E_y} = \binom{0}{1}$, respectively. This pair defines a simple basis that can be used to express the electric field for any wave.

a) Consider the E-field whose real part is $\vec{E} = 7\cos(kz - wt)\,\hat{\imath} - 5\sin(kz - wt)\,\hat{\jmath}$ (with implied units attached to the 7 and 5). Figure out how to decompose this field in terms of $\overrightarrow{E_x}$ and $\overrightarrow{E_y}$. By that

I mean find the A and B such that $\vec{E} = A\overrightarrow{E_x} + B\overrightarrow{E_y}$. You may find it helpful to start by expressing $\vec{E}$ in terms of a Jones vector. And don't forget that you can use imaginary coefficients if you need to.

b) An alternative basis for describing polarization is referred to as circular polarization. The basis vectors (in Jones notation) are $\overrightarrow{E_L} = \frac{1}{\sqrt{2}}\binom{1}{i}$ and $\overrightarrow{E_R} = \frac{1}{\sqrt{2}}\binom{1}{-i}$, describing left-circular and right-circular polarization respectively.

Explain, using an appropriate combination of words, equations, and diagrams, why this basis is referred to as *circular* polarization. It's no great challenge to find the answer on the interwebs, so make sure your explanation is strong and is uniquely your own. Also mention why those $\frac{1}{\sqrt{2}}$ factors are there.

c) Express the E-field from part (a) in terms of $\overrightarrow{E_L}$ and $\overrightarrow{E_R}$.

Two things to take away from this problem: 1) How to express polarization in general and 2) That circular polarization, while it sometimes sounds like an odd thing, is just another basis to work in, one that happens to come up a lot in optics labs.

3) (From Pollack and Stump 11.34) This problem gives us a look at the actual mechanical effects of fields on particles, and shows us how to treat the general problem numerically (real problems almost always have numerical solutions).

Consider an electromagnetic wave with vector potential $\vec{A}(\vec{x},t) = \hat{j}f(x-ct)$. (The scalar potential is 0.) The function $f(x-ct)$ approaches 0 as $x \to \pm\infty$, so the electromagnetic field is a wave packet. Suppose the wave hits an electron (charge –e, mass m) initially at rest at the origin.

a) Derive the equations of motion for the electron velocity components v_x, v_y, v_z.

b) Show that $v_z(t) = 0$.

c) Show that $v_y(t) = \left(\frac{e}{m}\right)f(x-ct)$, where x is the x position of the electron at time t.

d) Show that $v_x\left(1 - \frac{v_x}{2c}\right) = \frac{e^2}{2m^2c}[f(x-ct)]^2$

e) Describe in words and pictures the trajectory of the electron, assuming the wave packet has a short length. In particular, show that the electron will have a positive displacement in the x direction.

f) Assume $v_x \ll c$ and $f(\xi) = Ke^{-\xi^2/d^2}$. Use a computer to solve the differential equation for x(t) numerically and plot the result. (Let d be the unit of length, d/c the unit of time, and choose a relatively small value of the dimensionless constant $eK/(mc)$.)

g) Solve for y(t) and plot the trajectory in space. For example, in Mathematica, solve simultaneously the coupled parametric equations for x(t) and y(t) using NDSolve and make the plot with ParametricPlot.

Some clarifications/hints follow:

For c & d you have to be super careful with the difference between total and partial derivatives, and attend to the fact that f is a function of $(x - ct)$. It's tricky. Fortunately, it's a "show that" problem, so if you get stuck on c & d you can still do the rest. I found it useful to define $\eta = x - ct$ so as to write $f(x - ct)$ as $f(\eta)$. That helped me keep track of all the multivariate chain rule stuff. Though I still had to write things out in almost embarrassing detail to get all the terms to show up right.

For e I'm not really sure what pictures you could draw that aren't redundant with f & g, so if you're not sure what to draw, don't sweat it.

For f, it's implied that $\xi = x - ct$. Also, "relatively small" is kind of vague. Let's say "relatively small" is 0.1.

For f also note that if they say, for example, that d is a "unit" of length, that means d has unit value. Which is to say, it's 1. And if d is unit and d/c is unit, so must be c. Really, we're just trying to get rid of all the constants so we can zoom in on the qualitative behavior of the system.

4) (From Pollack and Stump 11.28)

The wave fronts of the spherical wave in Sec. 11.5.4 are spheres. However, the energy flux is not isotropic.

a) In the radiation zone show that the differential power $dP/d\Omega$, i.e., the average power per unit solid angle, is proportional to $\sin^2\theta$, where θ is the polar angle.

To do this, first show that $\frac{dP}{d\Omega} = r^2 \vec{S}_{avg} \cdot \hat{r}$ (this should be a fairly short proof). Then calculate that differential power in the radiation zone from the asymptotic fields.

b) Calculate the fraction of the total power within plus or minus 10 degrees of the equatorial plane. Compare your result to the fractional solid angle of that range of directions (i.e., its fraction of the total solid angle 4π).

c) So this "spherical wave" obviously doesn't have spherical symmetry. Something about the physical situation must break the symmetry if we don't get a spherically symmetric solution. What is it?

5) (From Pollack and Stump 13.23)

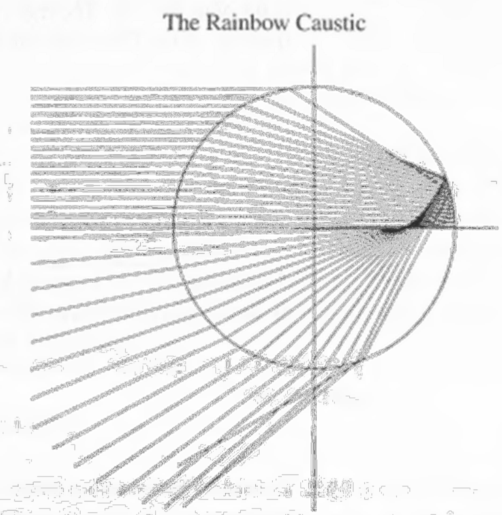

FIGURE 13.13 Illustration of how light refraction and reflection in a water droplet causes a rainbow. Light incident from the left is refracted on entering the spherical drop, reflects partially from the back surface, and is refracted on leaving the front surface. (Only rays incident above the midplane are shown.) The caustic—the region where the exiting rays are most concentrated—is the rainbow. The colors of the rainbow are the result of dispersion; the angle of the caustic varies with wavelength.

Figure 13.13 shows the light scattering process that creates the primary rainbow. Light rays at varying impact parameter refract into a spherical water drop, reflect from the back surface, and refract out of the drop. (In the figure, only the rays entering the upper half of the drop are shown. The rays shown are the rays that would reach the ground.) At a scattering angle of 42 degrees there is a concentration of scattered rays, called the caustic, and that somewhat more intense scattered light is the rainbow.

a) Explain why the ordering of colors (ROYGBIV) is red at the outer edge of the arc, and violet at the inner edge.

b) A secondary rainbow, in which the order of colors is reversed, is sometimes visible at a higher angle than the primary. Explain this second arc.

c) Explain why the area inside the primary rainbow is brighter than the area outside.

Note that “explain” doesn’t mean copy text from Wikipedia with a couple words changed. I want to see diagrams and geometry and stuff. Roughed out, at least.