PH311 Midterm answers and grading: 10/17/06

1 The time evolution of a spin 1/2 particle

You can do this either by power series expansion or an eigenvalue decomposition. E.g.,

$$\begin{aligned}\exp(i\omega t\mathbf{P}) &= 1 + (i\omega t\mathbf{P}) + \frac{1}{2}(i\omega t\mathbf{P})^2 \cdots \\ &= 1 + i\omega t\mathbf{P} - \frac{1}{2}\omega^2 t^2 \mathbf{P}^2 \cdots\end{aligned}$$

Show now that $\mathbf{P}^2 = I$ and hence all odd powers of P equal $\mathbf{P}$ and all even powers equal the identity. Thus the power series expansion of $\exp(i\omega t\mathbf{P})$ reduces to

$$\begin{aligned}& 1 - \frac{1}{2}\omega^2 t^2 \mathbf{I} + \frac{1}{24}\omega^4 t^4 \mathbf{I} \cdots + i\left(\omega t\mathbf{P} - \frac{1}{6}\omega^3 t^3 \mathbf{P} \cdots\right) \\ &= \left(1 - \frac{1}{2}\omega^2 t^2 + \frac{1}{24}\omega^4 t^4 \cdots\right)\mathbf{I} + i\left(\omega t - \frac{1}{6}\omega^3 t^3 \cdots\right)\mathbf{P} \\ &= \cos(\omega t)\mathbf{I} + \sin(\omega t)\mathbf{P}.\end{aligned}$$

2 The ladder operators

This is a straightforward exercise in matrix multiplication. You should all get this.

3 A model of radioactive decay

After 1 time interval the amount of decayed material is PA and amount of undecayed material is $(1-P)A$. The latter is available to decay during the second time interval. Hence after two time intervals the total amount of decayed material is $PA + P(1-P)A$ or $PA + PA(1-P)$. Continuing this process, after n time intervals the total amount of decayed material is:

$$PA + PA(1-P) + PA(1-P)^2 + ... + PA(1-P)^{(n-1)}$$

In the limit that $n \to \infty$, the sum of this geometric series is

$$\frac{PA}{1-(1-P)} = A$$

The physical interpretation of this is that eventually all the material decays.

4 Computing π

First note that:

$$\int_0^1 \frac{dx}{1+x^2} = [\arctan(x)]_0^1 = \frac{\pi}{4}$$

Next, expand the integrand in a power series and integrate term by term. This will allow you to write $\frac{\pi}{4}$ as a (slowly-convergent!) series.

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 \cdots$$

Integrating this term by term we have

$$x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

Evaluated at the lower limit (zero), this is zero. Hence the value of the integral is:

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \cdots$$

Hence,

$$\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \cdots\right)$$

In fact,

$$4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7}\right) = 2.895$$

If you take this out to 100 terms you get 3.16. This is a **terrible** way to compute π.