10/6/06

Note Title 10/6/2006

Midterm Review

Roel Snieder available next week: Monday 2-3

Tuesday all afternoon

Green Center 240 F

Exam will be 50 minutes

open book

3 out of 4 questions

No Least squares or ε-values (covered on next HW).

Practice questions

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} \lambda^2 & 0 \\ 0 & \lambda_2^2 \end{pmatrix}$$

$$\vdots$$

a) $$A^N = \begin{pmatrix} \lambda_1^N & 0 \\ 0 & \lambda_2^N \end{pmatrix}$$

$$\boxed{\begin{aligned} &A^T = A \\ &A = Q \Lambda Q^T \\ &A^2 = Q \Lambda \underbrace{Q^T Q} \Lambda Q^T \\ &\quad = Q \Lambda^2 Q^T \\ &A^N = Q \Lambda^N Q^T \end{aligned}}$$

b) $\ln(\underline{I} + A)$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} \ldots$$

Hence

$$\ln(I+A) = A - \frac{A^2}{2} + \frac{A^3}{3} - \frac{A^4}{4} \cdots$$

$$= \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} \lambda_1^3 & 0 \\ 0 & \lambda_2^3 \end{bmatrix} \cdots$$

$$= \begin{bmatrix} \lambda_1 - \frac{1}{2}\lambda_1^2 + \frac{1}{3}\lambda_1^3 + \cdots & 0 \\ 0 & \lambda_2 - \frac{1}{2}\lambda_2^2 + \frac{1}{3}\lambda_2^3 - \cdots \end{bmatrix}$$

$$= \begin{bmatrix} \ln(1+\lambda_1) & 0 \\ 0 & \ln(1+\lambda_2) \end{bmatrix}$$

- $$\lim_{x \to 0} \frac{1 - \cos x}{x^2}$$

$$= \lim_{x \to 0} \frac{1 - \left(1 - x^2/2 + x^4/24 \cdots\right)}{x^2}$$

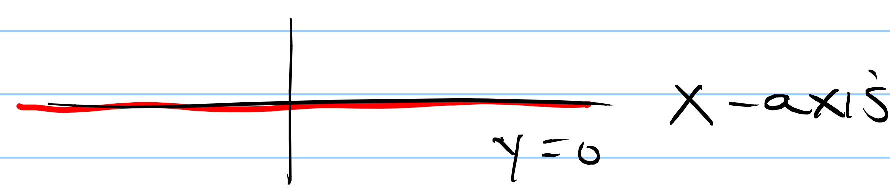
$$= \lim_{x \to 0} \frac{1}{2} - \frac{x^2}{24} \cdots$$

$$= \frac{1}{2}$$

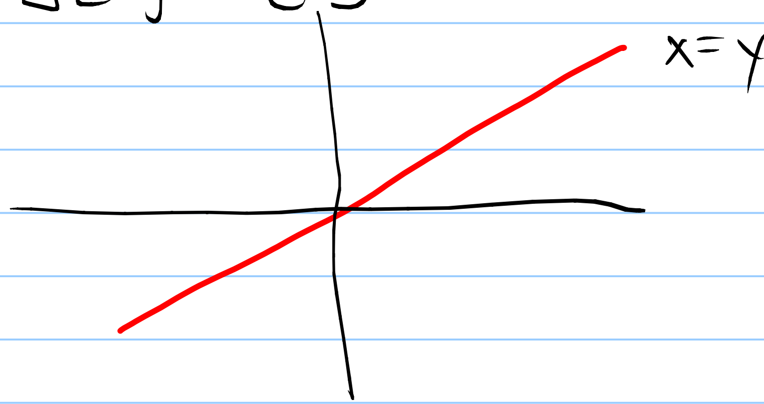
Null space of column space

$$\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

Column space = span $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$



$$\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \quad x = y$$



$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

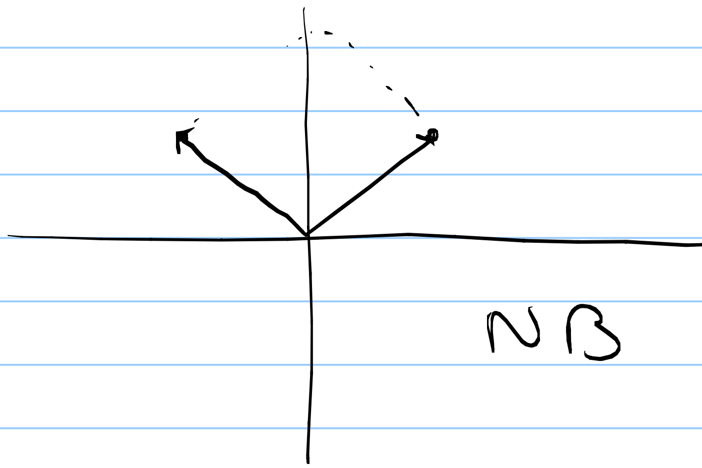
all of $\mathbb{R}^3$ is null space

column space $= \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Box problem

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$



rotation.

NB $\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

So $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ is an orthogonal matrix

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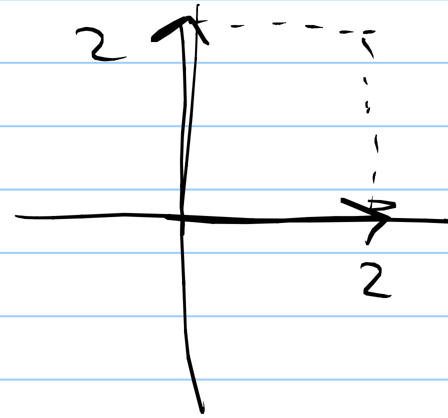
$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

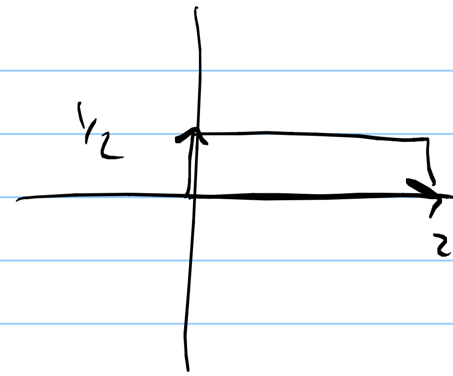
$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$



<u>stretch</u>

$$\begin{bmatrix} 2 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

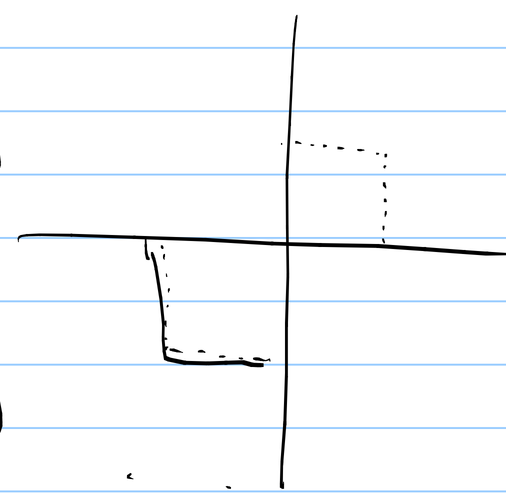
$$\begin{bmatrix} 2 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1/2 \end{bmatrix}$$



shrink in 1 Dim.
Stretch in the other

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

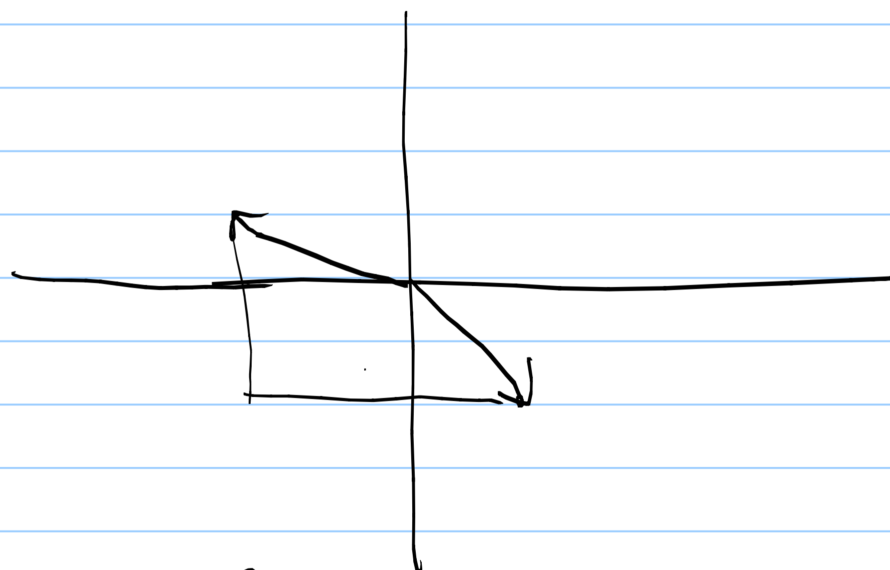


reflection about origin

So far all of these transf.
Have preserved the angle
between the vectors

$$\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$



$$\underbrace{\begin{bmatrix} -2, & 1 \end{bmatrix}} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = -2 - 2 = -4$$

$$\| \cdot \| = \sqrt{2^2 + 1^2} = \sqrt{5}$$

$$\cos \theta = \frac{-4}{5}$$

$$\frac{x}{(1-x)^2} = x + 2x^2 + 3x^3 + 4x^4 + \cdots$$

$$= \sum_{k=1}^{\infty} k x^k$$

Let $P = \sum_{k=m}^{N} x^k$

$= x^m + x^{m+1} + \cdots x^N$

Let $S_N = 1 + x + x^2 + \cdots x^N$

So $P = S_N - [\underbrace{1 + x + x^2 + \cdots x^{m-1}}_{S_{m-1}}]$

$$S_N = \frac{1 - x^{N+1}}{1 - x} \qquad S_{m-1} = \frac{1 - x^m}{1 - x}$$

$$\Rightarrow \quad P = \frac{1-x^{N+1}}{1-x} - \frac{1-x^m}{1-x}$$

$$= \boxed{\frac{x^m - x^{N+1}}{1-x}}$$

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$$

case 1) $\alpha = 1 \quad \beta = \gamma = 0$

$$x + y = 1 \qquad x = 1$$
$$y = 0 \qquad y = 0$$
$$2y = 0$$

So

$$1 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 0 \cdot \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

case 2 $\quad \alpha = \beta = 0 \quad \gamma = 1$

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \Rightarrow$$

$$\left.\begin{aligned} x + y &= 0 \\ y &= 0 \end{aligned}\right\} \quad x = y = 0$$

$$2y = 1 \quad \text{impossible}$$

therefore RHS not in column space.

$$A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$C = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

×	A	B	C	I
A	I	C	B	A
B	C	I	A	B
C	B	A	I	C
I	A	B	C	I

4-Group

$\vec{y} = T\vec{x}$ $\quad$ $\vec{x}, \vec{y}$ vectors

T matrix

given x + y is T determined uniquely?

$$Y_1 = T_{11} X_1 + T_{12} X_2 + T_{13} X_3$$

$$Y_2 = T_{21} \cdot X_1 + T_{22} X_2 + T_{23} X_3$$

$$Y_3 = T_{31} X_1 + T_{32} X_2 + T_{33} X_3$$

suppose

$$T_1 = \begin{bmatrix} T_{11} & & \\ & T_{22} & \\ & & T_{33} \end{bmatrix}$$

Then

$T_{11} = Y_1 / X_1$

$T_{22} = Y_2 / X_2$

$T_{33} = Y_3 / X_3$

This T_1 satisfies $\vec{Y} = T\vec{X}$

But how about

$$T_2 = \begin{bmatrix} & & T_{13} \\ & T_{22} & \\ T_{31} & & \end{bmatrix}$$

then
$$\begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} = \begin{bmatrix} & & T_{13} \\ & T_{22} & \\ T_{31} & & \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$$

$$\Rightarrow \quad T_{13} = Y_1 / X_3$$
$$T_{22} = Y_2 / X_2$$
$$T_{31} = Y_3 / X_1$$

So both T_1 + T_2 satisfy

$$\vec{y} = T\vec{x}$$

The consequence of this is that division by vectors is not defined in general

outstanding issues

McClaurin Series

Row/Col. space

Gaussian elimination

A A

$\vec{a}_i$ ith column of A

span$\{\vec{a}_i\}$ set of all linear comb. of $\{\vec{a}_i\}$

$\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ span$\left\{ \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix} \right\} = R^2$

column space $= R^2$

$\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$ column space x-axis

$\alpha \begin{bmatrix}1\\0\end{bmatrix} + \beta \begin{bmatrix}0\\0\end{bmatrix}$

$A\vec{x} = \vec{y}$ solution $\Leftrightarrow$ y is in the column space of A.

Gaussian

$$x + 4y = 8$$
$$3x - y = 1$$

$$3x - 1 = y$$
$$x + 4(3x - 1) = 8$$

$$\begin{bmatrix} x & x & x \\ x & x & x \end{bmatrix} \rightarrow \begin{bmatrix} 1 & & \\ 0 & 1 & \cdots \end{bmatrix}$$

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

$$\boxed{a_0 = f(0)}$$

$$f'(x) = a_1 + 2a_2 x + 3a_3 x^2$$

$$\boxed{f'(0) = a_1}$$

$$f''(x) = 2a_2 + 3 \cdot 2 \, a_3 x + \cdot$$

$$\boxed{\frac{f''(0)}{2} = a_2}$$

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

$$f(x) = f(c) + f'(0)\,x + \frac{f''(0)}{2} x^2 + \cdots$$

e^x, $\cos x$, $\sin$

$\frac{1}{1-x}$ $\quad \log(x)$