# Uncertainty Analysis

**Introduction.** An *error* is the difference between any single measurement and the true value of some *measurand*. An *uncertainty* is a mean error or an estimate of the mean error of a series of measurements.

There are 2 kinds of uncertainty: "measured" (called Type A by the International Organization for Standardization, or ISO) and "all others" (Type B). A measured uncertainty is the *standard deviation of the mean* of a set of measurements. The standard deviation you are familiar with is called the *sample* standard deviation and is given by

$$s = \sqrt{\frac{\Sigma\,(m_i - \mu)^2}{(N - 1)}}\ , \qquad (1)$$

where $m_i$ are the measured values, $\mu$ is the mean value, and $N$ is the number of measurements. The Excel function STDEV calculates $s$. If the errors are randomly distributed, then $s$ is approximately independent of $N$, and the measurement is said to be *stationary*.

The *standard uncertainty* is the standard deviation *of the mean*, or SDOM, and is given by

$$\sigma = s/\sqrt{N}\ . \qquad (2)$$

Equation (2) reflects that the more measurements we take, the more certain we are of the average value. Unfortunately, the square-root dependence reflects a law of diminishing returns; for example, to reduce the standard uncertainty (or uncertainty, for short) by a factor of 10, we have to take 100 times more measurements.

"All other" uncertainties are estimated or, for example, imported from manufacturer's specifications. To estimate an uncertainty, we first make an educated guess of the largest probable error. With a digital voltmeter, the largest probable error is 1/2 the *least count* of that voltmeter, presuming that the voltmeter is correctly calibrated. That is, if the least count of the voltmeter is 1 mV, then the largest probable error is 0.5 mV. (Why?)

Let us say that the least count of the voltmeter is 1 mV and we measure a voltage of 1.000 V. The true value of the voltage is between 0.9995 V and 1.0005 V, and any value within that range is equally probable. The probability distribution that describes the true value is therefore a rectangular function that is 0 outside the range between 0.9995 V and 1.0005 V. This range is called the *confidence interval* of the measurement, because we are completely confident that the true value lies somewhere within it (again, presuming that the voltmeter is precisely calibrated).

The largest probable error is, in this case, the half-width of the confidence interval. It is not the same as the uncertainty. To the contrary, the uncertainty is a standard deviation. The standard deviation of a rectangular distribution is its half-width divided by $\sqrt{3}$, so we *define* the standard uncertainty as

$$u_m = \Delta m/\sqrt{3}\ , \qquad (3)$$

where the subscript $m$ refers to the measurand and $\Delta m$ is the largest probable error of the measurand $m$. We do not divide by $\sqrt{N}$ , since this component of uncertainty cannot be reduced by averaging.

Both $u_m$ and $\sigma$ may be important in a given measurement. Indeed, there may be several *components of uncertainty* $u_1$, $u_2$, $u_3$,…, though there is often only 1 measured uncertainty $\sigma$. When several components of uncertainty are important, they are added *in quadrature*:

$$u_c=\sqrt{\sigma^2+u_1^2+u_2^2+u_3^2+\cdots} \ , \tag{4}$$

where $u_c$ is called the *combined standard uncertainty*. By convention, $u_c$ is a *positive* number.

The result of a measurement is expressed as

$$\mu \pm 2u_c \ , \tag{5}$$

where $\mu$ is the mean or measured value and $u_c$ is the combined standard uncertainty. The quantity $2u_c$ is called the *expanded uncertainty*. If $\mu$ is 3.2 m and $u_c$ is 0.1 m, then we state that the measured value is 3.2 ± 0.2 m (not ± 0.1 m). Note the correct number of significant digits; 3.200 ± 0.20 m, for example, is incorrect.

The interval $\pm 2u_c$ is the *95 % confidence interval*; that is, we think there is a 95 % probability that the true value lies within the range between $\mu-2u_c$ and $\mu+2u_c$ . In reality, cautious scientists and engineers often overestimate the components of uncertainty, so the probability is most likely much higher than 95 %. The interval between $\mu-3u_c$ and $\mu+3u_c$ is the *99.7 % confidence interval*; there is virtually no chance that the true value lies outside this interval if the errors are estimated accurately.

Save this thought for a later lab: Sometimes we need to measure one quantity in order to calculate the value of another quantity. For example, if we want to know the average velocity of a projectile, we might measure the time needed to traverse a given distance and then calculate the velocity from $v=d/t$ . If we know $d$ very accurately, then we need to relate the uncertainty of $v$ to the uncertainty of $t$. More generally, we might have $v=f(t)$ . Without going into detail, we express the largest possible error as the first term in a Taylor series:

$$\Delta v=\frac{\partial f}{\partial t}\Delta t \tag{6}$$

or, equivalently (because $u$'s and $\Delta$'s are related by $\sqrt{3}$ ),

$$u_v=\frac{\partial f}{\partial t}u_t \ . \tag{7}$$

A calculation based on Equation (6) or (7) is called *propagation of error* or *propagation of uncertainty*.

**Experiment** (team activity).

1. Study the micrometer checked out to you by the instructor. Devise a strategy to measure the mean diameter of the pencil. For this exercise, let us define the diameter as the distance between two opposed apexes (as opposed to the distance between two flat facets). Do not try to interpolate between the least-count markings of the micrometer. Most instruments are assumed to be no more accurate than their least counts; in the case of the micrometer, play and error in the *pitch* of the threads probably limit its accuracy to the least count. The calibration of a digital voltmeter is likewise no better than its least count implies and may in fact be poorer because of nonlinearity in the electronics.

2. Calculate the mean $\mu$ of your measurements. Calculate the standard deviation of the mean,

$$\sigma = \sqrt{\frac{\Sigma\,(m_i - \mu\,)^2}{N(N-1)}}\ ,$$

where $m_i$ are the individual measurements and $N$ is the number of measurements. You may use the Excel function STDEV and divide the result by $\sqrt{(N)}$. A note that standard deviation has little meaning when $N$ is less than perhaps 10.

3. Calculate the standard uncertainty $u_{lc}$ due to the least count (lc) of the micrometer. Compare $u_{lc}$ with the measured uncertainty $\sigma$. How many measurements would you have to make to reduce $\sigma$ to a value equal to $u_{lc}$? Many measurements are considered accurate when the random uncertainty $\sigma$ is equal to the estimated uncertainty due to the instrument itself, as noted in 1.

4. Suppose that the pencil is made of wood. Suppose further that the temperature in some application might be between 0 and 40 °C, and the temperature in the lab right now is approximately 20 °C. The coefficient of expansion of wood along the grain is 3 to $4.5\times10^{-6}$/K; across the grain, it is 25 to $40\times10^{-6}$/K and depends on specific gravity. If the diameter of the pencil is $D$, estimate the largest probable error $\Delta D$ that results from thermal expansion. What is the corresponding standard uncertainty $u_T$ of the measured diameter? (Hint: It is conventional to assume, conservatively, that estimated uncertainties may be described by a rectangular distribution. Hence, calculate the largest probable error of diameter due to thermal expansion and divide it by $\sqrt{3}$ .)

5. We have identified 3 components of uncertainty: the least count of the micrometer, the temperature, and the random error due to the irregularity of the pencil. Are any of the components negligible? Add the non-negligible components in quadrature and calculate the combined standard uncertainty $u_c$. Evaluate $u_c$ and calculate the expanded uncertainty 2 $u_c$.

6. State the mean thickness of the pencil and the expanded uncertainty in the correct form and with the right number of significant digits.

7. Suppose that the pencils must be pushed snugly through a hollow cylinder, much as an optical fiber is passed through a *ferrule* for making connections. We want to have good confidence that any pencil will fit through any cylinder. What must be the minimum diameter of the cylinder? Hint: Do you use the SDOM or the sample standard deviation? Why? Assume that the inside diameter of the cylinder is known accurately.