MATH348: AEM – WEEKLY WORK RECORD NAME: Scott Strong

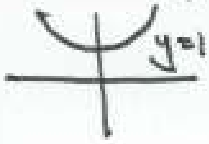
1. Weekly Log – Record all intervals of AEM work outside of class.

Date	Tasks	Time Spent
8/29	Detailed derivations to HW1 prob 1-2. Scanned + put online	3-6pm

2. Questions – List of questions about your work. Direct grader to your work; post-its are nice.

3. Notes – List anything you may want to quickly reference in the future.

$2\cosh(x) = e^{x} + e^{-x}$, $2\sinh(x) = e^{x} - e^{-x}$, $e^{ix} = \cos(x) + i\sin(x)$



$2\cos(x) = e^{ix} + e^{-ix}$

$2i\sin(x) = e^{ix} - e^{-ix}$

4. Comments – Anything else you think of along the way.

It will pay to really understand 1, 2!

8/29/2012

MATH348: SPRING 2012 - HOMEWORK 1

RESULTS FROM ORDINARY DIFFERENTIAL EQUATIONS

Some light being pulled you up from nights party and said clap your hands if you think your soul is free.

ABSTRACT. With the exception of linear algebra, most of our course material will be dominated by results from ordinary differential equations. So, it would be a good idea to solidify and provide scope to our understanding of linear 2^{nd}-order ordinary differential equations for which we should note that our goal has always been to find two linearly independent solutions whose linear combination is then added to a particular solution to form the general solution set. If this is the prescription then the *real* trouble is in how one finds each of the pieces, that is, the two linearly independent solutions to the homogeneous problem and one particular solution to the inhomogeneous problem. These problems take us through the standard procedures and can be summarized as follows:

P1. In this problem we go back to the fundamentals. Since you have been trained to do constant coefficient problems, we take a little time to remind ourselves of the mass-spring models and how the quadratic equation can be used to define particular types of mass-spring dynamics. Also, we introduce the hyperbolic trigonometric functions and cast solutions to the mass-spring ODE into hyperbolic form.

P2. Building on the 'new' hyperbolic trigonometric functions and in preparation of things to come we breakdown the undamped oscillator case. To really drive home the inter-relationships between the hyperbolic trigonometric functions to their circular counterparts we do the problem through power-series methods, which reminds us of the method and some very important Taylor series formulae. Most of this I expect you have seen so we finish with the concept of a boundary value problem, which will be important for our study of PDE.

P3. A boundary value problem is a ubiquitous thing and we are not always lucky enough to have an undamped oscillator as the underlying ODE. However, there types of ODEs that are common to BVP and linear PDE have a general form known as Sturm-Liouville form. In this problem we look at the different types of ODE that come from such an equation.

P4. We didn't need to do problem 2 using power-series, it was just good practice and an important reminder of common Taylor series. We should spend time with a problem that requires power-series techniques. Since power-series methods comprise a large amount of material we trim our study down to a specific equation, Bessel's Equation, that can be dealt with through the techniques we have just practiced and those taught in MATH225. The solution, known as Bessel's function of the first-kind, will be given character later when we study the vibrations of an ideal drumhead.

P5. Lastly, we can't leave this material without showing some important, though cumbersome, formulae that allow us, given one solution to a second order linear ODE, to construct the general solution. While we won't need these formulae again, they do justify the guessing and checking you were taught in previous classes, which some of you may have thought was a little fishy.

$y(x) = y_h(x) + y_p(x)$

Where $y_h(x)$ solves

$a(x)y'' + b(x)y' + c(x)y = 0$

1. REVIEW OF 2^{nd}-ORDER CONSTANT LINEAR HOMOGENEOUS ODEs

The ordinary differential equation (ODE),

$$ay'' + by' + cy = 0, \quad a, b, c \in \mathbb{R} \tag{1.1}$$

is called a homogeneous linear ODE with constant coefficients and defines an infinite family of solutions with two degrees of freedom. Specifying a value of the solution and its derivative for the same point in time/space determines a single unique solution from the infinite family. A more

Date: January 10, 2012.

physical way of thinking about this is to write (1.1) as,

$$my'' + by' + ky = 0, \quad m, b, k \in \mathbb{R}^+, \tag{1.2}$$

which models the *free* dynamics of a point-mass attached to an Hookean spring moving in a viscous background. To find unique dynamics one must know the initial displacement from equilibrium, $y(0) = y_0 \in \mathbb{R}$, and the initial velocity, $y'(0) = y_0' \in \mathbb{R}$. To build intuition from your previous coursework, namely MATH225 and PHGN100, we continue with the notation of (1.2).

1.1. **Quadratic equations and its three cases.** [1] Solve (1.2) three times with the following values

$$(m, b, k) = \{(2, 8, 6), (1, -4, 13), (1, 4, 4)\} \tag{1.3}$$

and initial conditions $y(0) = 1$ and $y'(0) = -1$.

1.2. **The role of the coefficient of kinetic friction.** [2] Let $m = 1, k = 9$ and determine values for b such that the system is undamped, underdamped, critically damped and overdamped.

1.3. **The general case.** Show that for arbitrary $m, b, k \in \mathbb{R}$ the general solutions are given by:[3]

[1] You may go directly to problem 1.3 if this review is not needed.

[2] Again, you may skip directly to problem 1.3 if this is clear to you.

[3] This table summarizes all the different cases for constant coefficient problems and though there are a lot of symbols it isn't too hard to come up with them. The only real tricky parts are the conversions from exponentials to circular/hyperbolic trigonometric functions. I offer the following hint/algorithm to work this table out.

i. Begin with assuming that $y(t) = e^{\lambda t}$ to get the characteristic polynomial $m\lambda^2 + b\lambda + k = 0$ whose solution is given by the quadratic equation – see first-row last column of the table.
ii. Notice that if $b^2 - 4km \neq 0$ that there are two different exponential solutions for each decay-rate/frequency λ_1, λ_2 and that these two solutions are linearly independent by a nonzero Wronskian determinant, $W = \det\left(\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}\right) = y_1 y_2' - y_1' y_2$, which is easiest to do in exponential form.
iii. Using the substitutions given in the right-hand columns, show that the exponential forms are equivalent to their 'trigonometric' counterparts. Linear independence should still be maintained since this is just an algebraic recasting, i.e. addition and scalar multiplication.
iv. In the repeated root case, show that $y_2(t) = te^{\lambda t}$ is a solution to the original ODE and that this is linearly independent to y_1.

After this you are done.

Discriminant	Solutions	Homogeneous Solution	Definitions
Overdampped $b^2 - 4mk > 0$	$y_1(t) = e^{\lambda_1 t}$ $y_2(t) = e^{\lambda_2 t}$	$y_h(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$ $= a_1 e^{\alpha t}\cosh(\beta t) + a_2 e^{\alpha t}\sinh(\beta t)$	$c_1, c_2, a_1, a_2 \in \mathbb{C}$ $2c_1 = a_1 + a_2,\ 2c_2 = a_1 - a_2,$ $\lambda = \alpha \pm \beta$ $\alpha = \frac{-b}{2m},\ \beta = \frac{\sqrt{b^2 - 4km}}{2m}$ $\lambda_1 = \frac{-b + \sqrt{b^2 - 4km}}{2m}$ $\lambda_2 = \frac{-b - \sqrt{b^2 - 4km}}{2m}$
Underdampped $b^2 - 4mk < 0$	$y_1(t) = e^{\lambda_1 t}$ $y_2(t) = e^{\lambda_2 t}$	$y_h(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$ $= a_1 e^{\alpha t}\cos(\beta t) + a_2 e^{\alpha t}\sin(\beta t)$	$c_1, c_2, a_1, a_2 \in \mathbb{C}$ $a_1 = c_1 + ic_2,\ a_2 = c_1 - ic_2$ $\lambda = \alpha \pm \beta i$ $\alpha = \frac{-b}{2m},\ \beta = \frac{\sqrt{4km - b^2}}{2m}$
Critically damped $b^2 - 4mk = 0$	$y_1(t) = e^{\lambda t}$ $y_2(t) = te^{\lambda t}$	$y_h(t) = c_1 e^{\lambda t} + c_2 t e^{\lambda t}$	$c_1, c_2 \in \mathbb{C}$ $\lambda = \frac{-b}{2m}$

$b=0 \Rightarrow$ undampped

Notice: $-\frac{b}{2m} < 0$ for a mass-spring system. Thus, all soln decay.

2. Power Series, 'Trigonometric' Functions and Boundary Conditions

One way to think about a function is by the differential equation that defines it. For instance, the exponential function we all know and love can be thought of as the unique function that solves the equation $y' = ay$, where a is some scalar, which is to say that the exponential function is the only function that has the property that one differentiation returns a constant multiple of itself. It is natural to then seek functions that do this upon two differentiations. That is, we seek to find functions that obey the equation,

$$y'' + \lambda y = 0, \quad \lambda \in \mathbb{R}, \tag{2.1}$$

which happens to be an equation that we will see repeatedly through the course.

2.1. **Intuition.** What functions satisfy this equation? Be careful to consider this for all possible values of λ. See class notes on 8.29.12.

2.2. **Power Series Solution to ODE.** It is instructive to solve the previous problem by power series. First, it is good practice and second it helps to connect the previous circular/hyperbolic trigonometric functions to one another and their exponential 'parent'. Find the solution to (2.1) by assuming $y(x) = \sum_{n=0}^{\infty} a_n x^n$ and finding a general formula a_n and thus a solution $y(x)$.[4] Make sure to consider your solution for all possible values of λ.

2.3. **Introduction to Boundary-Value Problem.** You may have noticed that we have switched our independent variable from time to space. This is because ODEs, while often associated with mechanics/dynamics, often arise from spatial problems. However, in the spatial setting the isn't really the notion of an initial state. Consequently, one is typically provided spatial conditions called boundary conditions that prescribe the value of a solution or its slope at two different points in space. Specifically,

$$l_1 y(0) + l_2 y'(0) = 0, \tag{2.2}$$

$$r_1 y(L) + r_2 y'(L) = 0. \tag{2.3}$$

This problem is intractable, by hand, for general values of l_1, l_2, r_1, r_2. However, the following set of values,

	l_1	l_2	r_1	r_2
Case I	1	0	1	0
Case II	0	1	0	1
Case III	1	0	0	1
Case IV	0	1	1	0

lead to BVP that can be solved by hand. Find all nontrivial (non-zero) solutions and their corresponding eigenvalues to (2.1) that satisfy the previous boundary conditions.

3. Introduction to Sturm-Liouville Problems

The previous problem is a specific case of a more general type of differential equation that that underpins linear partial differential equations. First, we should notice that the previous problem is a type of eigenvalue problem, $\frac{d^2}{dx^2}[y] = -\lambda y$, which asks us to find the function y such that the differentiation transforms y to be a scalar multiple of itself.[5] With this in mind we define L, which takes in a function y and returns a linear differential transformation of it in the following way.

$$L[y] = \frac{1}{w(x)}\left(-\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + q(x)y\right). \tag{3.1}$$

The corresponding eigenproblem, $L[y] = \lambda y$, with the boundary conditions

$$l_1 y(0) + l_2 y'(0) = 0, \tag{3.2}$$

$$r_1 y(L) + r_2 y'(L) = 0. \tag{3.3}$$

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[4] Recall that any time one guesses a solution to a differential equation it is possible to check this guess by direct substitution. The program is this:

i. From the power series find a formula for y' and write the differential equation.
ii. Using re-indexing, write the differential equation as one summation – this can't always be done for all terms but it should be your primary goal of all power-series solution problems. You should have something that looks like $\sum [?]\, x^n = 0$ and since power functions are linearly independent the only way to guarantee this sum is zero is by forcing the coefficient to be zero. That is, the only way a linear combination of linearly independent objects can be zero is to require that they are all scaled to zero, $[?] = 0$.
iii. Setting the previous coefficients to zero gives a recurrence relation and the goal now is to find a solution. This is typically achieved, if possible, by plugging in values for the index and through using previous coefficients find a pattern.
iv. If a general formula is found then the result is plugged back into the original power series and, when possible, find any known Taylor series.

[5] Recall from differential equations the matrix eigenvalue/eigenvector problem that looked like $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$ and asks you to find a vector $\mathbf{x}$ such that transformation by the matrix multiplication of $\mathbf{A}$ returns a scalar multiple of itself.

is called a Sturm-Liouville problem (SLP).[6] If p, p' and q are continuous functions on the interval $[0, L]$ and $p(x) \neq 0$ for all $x \in [0, L]$ then (3.1)-(3.3) is called a *regular* SLP. Otherwise it is called *singular*.

3.1. **'Standard' SLP.** Find the form of $L[y] = \lambda y$ where $p(x) = 1$, $q(x) = 0$, $w(x) = 1$. For what $x-$values is the problem singular?

3.2. **Bessel's Equation of order λ.** Find the SL ODE for $p(x) = -q(x) = [-w(x)]^{-1} = x$. For what $x-$values is the problem singular?

3.3. **Legendre's Equation.** Find the SL ODE for $p(x) = (1-x)^2$, $q(x) = 0$, $w(x) = 1$. For what $x-$values is the problem singular?

4. Introduction to Bessel's Equation

Regular SLP can be solved through standard power-series methods. On the other hand, singular SLP can be difficult and require more the more advanced Frobenius method. While Bessel's equation is singular at $x = 0$, for $\lambda = 0$ Frobenius' method actually reduces to the standard power-series method. That is, Bessel's equation of order-zero,

$$x^2 y'' + xy' + x^2 y = 0, \tag{4.1}$$

can be solved by assuming $y(x) = \sum_{n=0}^{\infty} a_n x^n$.

4.1. **Power-Series: Step I.** Using the power-series assumption, find the recurrence relation for (4.1), $a_{n+2} = -\dfrac{a_n}{(n+2)^2}$ and show that $a_{2n+1} = 0$ for $n = 0, 1, 2, 3, \ldots$.

4.2. **Power-Series: Step II.** Concentrating now on the even coefficients, show that $a_{2k} = \dfrac{(-1)^k}{2^{2k}(k!)^2} a_0$ and setting $a_0 = 1$ write down $J_0(x) = y(x)$, which is called *Bessel's function of the first-kind of order-zero*.[7]

5. General Theory of 2^{nd}–Order Linear ODE

When an ODE is linear and has constant coefficients then the problem is really asking us to find a function that returns multiples of itself upon successive differentiation. We know these functions to be exponentials/trig/hyperbolics and so the solution is some linear combination of these functions where the decay-rate/frequency is defined by a polynomial root finding problem. If we are not so lucky and the linear ODE has variable coefficients then we have the harder power-seres/recursion problem but what we should notice is that we haven't had to integrate anything. That is, we are able to solve ODEs without using integration, which is quite surprising. That is not to say that a theory of solving linear ODE through the use of integrals doesn't exist. In fact, if we are given just one solution to

$$a(x)y'' + b(x)y' + c(x)y = f(x) \tag{5.1}$$

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[6] SLP are interesting because it can be shown that they admit infinitely-many real eigenvalues that that increase to infinite. Moreover, the functions associated with distinct eigenvalues form a complete orthonormal set, which is to say that can be used as a basis for a particular set/space of functions. These are exactly the functions we will study in Fourier series and PDE.

[7] The reason that it is called of the first kind is because there should be two linearly independent solutions to the second order linear ODE and we have only found one. It turns out that we won't need the Bessel function of the second kind. So, we can stop here.

then we can find a second linearly-independent solution and a particular solution, [8] and thus the general solution, through the use of integrals.[9]

5.1. Second Linearly Independent Solution. Suppose that $a(x) = 1$, $b(x) = 4$, $c(x) = 4$, $f(x) = e^{-2x}$. [10] We know that $y_1(x) = e^{-2x}$ is a solution to this problem and using the formula,

$$y_2(x) = k(x)y_1(x),\ k(x) = \int \frac{p(x)}{[y_1(x)]^2}dx,\ p(x) = e^{-\int (b(x)/a(x))dx}, \tag{5.2}$$

it is possible to find a second linearly independent solution to the ODE.[11]

5.2. Particular Solution: Part I. Using the formula,

$$y_p(x) = y_2 \int \frac{f(x)y_1(x)}{a(x)W(x)}dx - y_1 \int \frac{f(x)y_2(x)}{a(x)W(x)}dx,\ W(x) = y_1(x)y_2'(x) - y_1'(x)y_2(x), \tag{5.3}$$

find the form for the particular solution for $a(x) = 1$, $b(x) = 4$, $c(x) = 4$, $f(x) = e^{-2x}$. [12] Also, verify this solution using the method of undetermined coefficients.

5.3. Particular Solution: Part II. With our newfound trust, we use the previous formula on a problem that couldn't have been analyzed through previous methods. Solve the previous ODE where $a(x) = 1$, $b(x) = 0$, $c(x) = 1$, $f(x) = \sec(x)$, where $x > 0$.

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[8] Recall that the idea was that given $a(x)y'' + b(x)y' + c(x)y = f(x)$ we seek two solutions $y_1(x)$ and $y_2(x)$ such that $y_1(x) \neq cy_2(x)$ to the case where $f(x) = 0$ for all x, i.e. two linearly independent homogeneous solutions, and then one solution to the full(non-homogeneous) problem. If this is done then the general solution (AKA all solutions) is written as $y(x) = y_h(x) + y_p(x)$ where $y_h(x) = c_1y_1(x) + c_2y_2(x)$ is the homogeneous solution and $y_p(x)$ is a particular solution to the whole problem.

[9] This is rarely needed since in most cases its possible to just always guess the right answers. However, it does come in handy if you are in a bind.

[10] This problem is degenerate in the sense that it contains a repeated eigenvalue. Worse, the inhomogeneous term competes with the associated eigenfunction. You can solve this completely using techniques from your previous course work. We will use some formula to justify these techniques.

[11] These formulae are found by assuming a solution of the form y_2 and checking it against the ODE. The steps are tedious but the result is general and shows that when they told you to use $y_2(x) = xy_1(x)$, they were right!

[12] You might notice that this can be done via the method of undetermined coefficients, which is considerably easier even if you have to multiply your 'guess' by two factors of x!

8/29/12

1) Constant Linear Homogeneous ODE

<u>The general case</u>:

Let $a,b,c \in \mathbb{R}$ then for $y(x) = e^{rx}$

$$ay'' + by' + cy = ar^2 e^{rx} + bre^{rx} + ce^{rx} =$$

$$= e^{rx}(ar^2 + br + c) = 0$$

Since $e^{rx} \neq 0$
$\Rightarrow$

$$ar^2 + br + c = 0 \Rightarrow r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- If $b^2 - 4ac < 0$ then [Notice: $4ac - b^2 > 0$

$$r = \frac{-b}{2a} \pm \frac{\sqrt{-1(4ac-b^2)}}{2a} = \frac{-b}{2a} \pm \frac{\sqrt{4ac-b^2}}{2a} i$$

$$= \alpha + \beta i$$

and thus for $k_1, k_2 \in \mathbb{R}$

$$e^{ix} = \cos(x) + i\sin(x)$$

$$y(x) = k_1 e^{(\alpha+\beta i)x} + k_2 e^{(\alpha - \beta i)x} =$$

$$= e^{\alpha x}\left(k_1 [\cos(\beta x) + i\sin(\beta x)] + k_2[\cos(-\beta x) + \sin(-\beta x)]\right)$$

$$\overset{*}{=} (k_1+k_2)e^{\alpha x}\cos(\beta x) + (k_1-k_2)i \cdot e^{\alpha x}\sin(\beta x) =$$

$$\overset{(**)}{=} \tilde{k}_1 e^{\alpha x}\cos(\beta x) + \tilde{k}_2 e^{\alpha x}\sin(\beta x), \quad \tilde{k}_1, \tilde{k}_2 \in \mathbb{R}$$

Notes:

(*) $\cos(-x) = \cos(x)$ } cosine is Even

$\sin(-x) = -\sin(x)$ } sin is odd

(**) $k_1+k_2 = \tilde{k}_1$, $(k_1-k_2)i = \tilde{k}_2$

- If $b^2-4ac=0$ then $r = -\frac{b}{2a}$

and

$$y(x) = k_1 e^{-\frac{b}{2a}x} + k_2 x e^{-\frac{b}{2a}x}$$

[~~She~~ See problem five for details of "proof" but remember how you were taught to guess.

- If $b^2-4ac>0$ then

$r = \alpha \pm \gamma$, $\gamma = \frac{\sqrt{b^2-4ac}}{2a}$

and

$$y(x) = k_1 e^{(\alpha+\gamma)x} + k_2 e^{(\alpha-\gamma)x} =$$

$$= e^{\alpha x}\left(k_1 e^{\gamma x} + k_2 e^{-\gamma x}\right) \overset{(*)}{=}$$

$$= e^{\alpha x}\left(\left[\frac{\tilde{k}_1 + \tilde{k}_2}{2}\right] e^{\gamma x} + \left[\frac{\tilde{k}_1 - \tilde{k}_2}{2}\right] e^{-\gamma x}\right) =$$

$$= e^{\alpha x}\left(\tilde{k}_1\left[\frac{e^{\gamma x} + e^{-\gamma x}}{2}\right] + \tilde{k}_2\left[\frac{e^{\gamma x} - e^{-\gamma x}}{2}\right]\right)$$

$$= \tilde{k}_1 e^{\alpha x}\cosh(\gamma x) + \tilde{k}_2 e^{\alpha x}\sinh(\gamma x)$$

<u>Note:</u>

(*) Just like $k_1 + k_2 = \tilde{k}_1$ can be done b/c there were #'s we didn't know in the first place, we "break up" k_1, k_2 into other unknown constants.

2) Power - Series + trig fn

Assume $y(x) = \sum_{n=0}^{\infty} a_n x^n$ then

$$y'' + \lambda y = \sum_{\cancel{n=0}}^{\infty} a_n \cdot (n)(n-1) x^{n-2} + \lambda \sum_{n=0}^{\infty} a_n x^n =$$

$n=2$ since $n=1,0$ are trivial terms

$$= \sum_{n=0}^{\infty} \underbrace{a_{n+2} (n+2)(n+1)\, x^n}_{} + \lambda a_n x^n =$$

$n=0 \rightarrow$ Shifted series forward in n to match 2nd series

$$= \sum_{n=0}^{\infty} x^n \left[a_{n+2}(n+2)(n+1) + \lambda a_n \right] = 0 \quad (!!)$$

Noting that $\rightarrow \det\left(\begin{bmatrix} x^n & x^{n+1} \\ n x^n & (n+1) x^{n+1} \end{bmatrix} \right) = (n+1) x^{2n+1} - n x^{2n+1}$
implies that $= x^{2n+1} \neq 0$, for all n

power fn are linearly independent except at $x=0$
which means the only way to get eqn (!!)
to be true for all x is by Req. that

$$a_{n+2} = \frac{-\lambda a_n}{(n+2)(n+1)}, \quad n = 0, 1, 2, \ldots$$

If $\lambda = 0$ then $a_{n+2} = 0$ for $n = 0, 1, 2, \dots$ which tells us nothing about a_0, a_1. Thus, $a_0, a_1 \in \mathbb{R}$ and we have from our assumption

Yay!

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 x^0 + a_1 x^1 = a_0 + a_1 x$$

If $\lambda > 0$ then

$n=0$, $a_2 = \dfrac{-\lambda a_0}{2 \cdot 1}$

$n=1$, $a_3 = \dfrac{-\lambda a_1}{3 \cdot 2 \cdot 1}$

$n=2$, $a_4 = \dfrac{-\lambda a_4}{4 \cdot 3} = \dfrac{-\lambda \cancel{a_0} (-\lambda) \cdot a_0}{4 \cdot 3 \cdot 2 \cdot 1}$

$\vdots$

$n=3$, $a_5 = \dfrac{-\lambda a_3}{5 \cdot \cancel{3} \cdot 4} = \dfrac{-\lambda}{5 \cdot 4} \cdot \dfrac{(-\lambda) a_0}{3 \cdot 2 \cdot 1} = \dfrac{\lambda^2 a_0}{5!}$

$\vdots$

$n = 2k$, $k = 0, 1, 2, \dots$

$$a_{2k} = \frac{(-1)^k a_0}{(2k)!} \cdot \lambda^k$$

$n = 2k+1$, $a_{2k+1} = \dfrac{(-1)^k \lambda^k a_1}{(2k+1)!}$

Thus, from our assumption we have

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = \overbrace{\sum_{n=0}^{\infty} a_{2n} x^{2n} + \sum_{n=0}^{\infty} a_{2n+1} x^{2n+1}}^{\text{Split into Even + odd terms}} =$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \lambda^n x^{2n}}{(2n)!} a_0 + \sum_{n=0}^{\infty} \frac{(-1)^n \lambda^n a_1 x^{2n+1}}{(2n+1)!} =$$

$$\lambda^{\frac{2n+1}{2}} = \lambda^n \lambda^{1/2}$$

$$= a_0 \sum_{n=0}^{\infty} \frac{(-1)^n (\lambda^{1/2} x)^{2n}}{(2n)!} + \underbrace{\frac{a_1}{\lambda^{1/2}}}_{\tilde{a}_1} \sum_{n=0}^{\infty} \frac{(-1)^n (\lambda^{1/2} x)^{2n+1}}{(2n+1)!} \quad (\star)$$

tricky!

$$= a_0 \cos(\sqrt{\lambda}\, x) + \tilde{a}_1 \sin(\sqrt{\lambda}\, x), \text{ yay!}$$

If $\lambda < 0$ then, (i) $a_{n+2} = \dfrac{|\lambda| a_n}{(n+2)(n+1)}$, $n = 0, 1, 2 \ldots$

(ii)

$$a_{2k} = \frac{\lambda^k a_0}{(2k)!}, \quad a_{2k+1} = \frac{\lambda^k a_1}{(2k+1)!},$$

The sign alternations go away!

and from $(\star)$

$$y(x) = a_0 \sum_{n=0}^{\infty} \frac{(|\lambda|^{1/2} x)^{2n}}{(2n)!} + \tilde{a}_1 \sum_{n=0}^{\infty} \frac{(|\lambda|^{1/2} x)^{2n+1}}{(2n+1)!} = a_0 \cosh(\sqrt{|\lambda|}\, x) + \tilde{a}_1 \sinh(\sqrt{|\lambda|}\, x)$$

Some interesting points:

- $\cosh(x) + \sinh(x) =$

$$= \underbrace{\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}}_{\text{Even}} + \underbrace{\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}}_{\text{odd}} = \overbrace{\sum_{n=0}^{\infty} \frac{x^n}{n!}}^{\text{all}} = e^x$$

- cosh ~~are~~ is the Even power terms from e^x [sinh are the odd ones]

- cosine then are the Even terms of e^x ~~where~~ with an imposed sign alteration. [Sine are the odds with sign alt.

__Point!__ This is the function of $i = \sqrt{-1}$

$$e^{ix} = \underbrace{\sum_{n=0}^{\infty} \frac{(ix)^{2n}}{(2n)!}}_{} + \sum_{n=0}^{\infty} \frac{(ix)^{2n+1}}{(2n+1)!} = \cos(x) + i\sin(x)$$

$$\frac{i^{2n} x^{2n}}{(2n)!} = \frac{(i^2)^n x^{2n}}{(2n)!} = \frac{(-1)^n x^{2n}}{(2n)!}$$

* Make Everyother Even term negative.

Lastly,

- $\cosh(ix) = \cos(x)$, $\sinh(ix) = i\sin(x)$.

If $\lambda < 0$

- $$y'' + \lambda y = 0 \iff E = \frac{(y')^2}{2} + \frac{\lambda y^2}{2} \Rightarrow$$

$E \neq 0$

$$\Rightarrow \underbrace{\frac{(y')^2}{2E} - \frac{|\lambda| y^2}{2E} = 1}$$

this is the Eqn of a Hyperbola in Phase Space!

See lecture notes from 1.18.2012 for graphs.

Boundary Value Problems:

· Fall 2012: we will Explore this in the context of PDE. It might not make a ton of sense now.

ODEs in DiffEQ class are typically given with initial values. This is b/c dynamics are made unique by

~~given a dynam~~ ~~and initial~~

(① Newton's Law ② Kirchoff's Law)

providing initial ① displacement ② Change and ① velocity ② Current for a dynamical rule like ① Newton's laws. ② Kirchoff's laws.

However, an ode can be spatial in nature and at this case one is typically given conditions at spatial boundaries.

Example 1: (Case I from table)

$$y'' + \lambda y = 0, \ \lambda \in \mathbb{R} \qquad (1)$$

such that

$$y(0) = 0, \ y(L) = 0 \qquad (2)$$

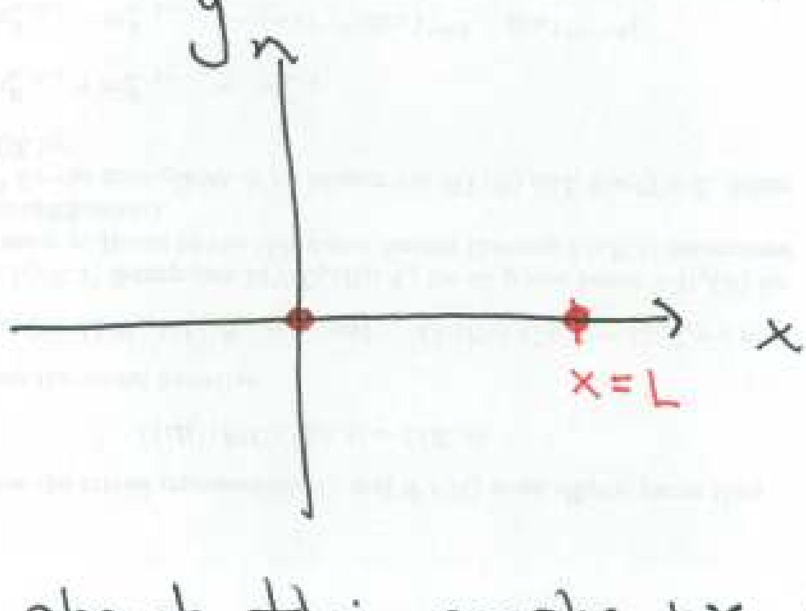
Recall that all soln to (1) are given by

$$\lambda > 0: \quad y_1(x) = c_1 \overset{\sin}{\cancel{\cos}}(\sqrt{\lambda}\, x) + c_2 \cos(\sqrt{\lambda}\, x)$$

$$\lambda < 0: \quad y_2(x) = c_3 \sinh(\sqrt{|\lambda|}\, x) + c_4 \cosh(\sqrt{|\lambda|}\, x)$$

$$\lambda = 0: \quad y_3(x) = c_5\, x + c_6$$

Now Eqn (2) mandates that of these soln we must choose those that pass through ~~the~~ both red points



Thinking about their graphs we conclude that

$$c_6 = c_5 = c_4 = c_3 = c_2 = 0$$

that is $\sin(\sqrt{\lambda}x)$ is the only fn that can pass through both points.

Thus,

$$y(L) = C_1 \sin(\sqrt{\lambda} L) = 0$$

$$\underbrace{\Rightarrow}_{\text{for } C_1 \neq 0} \sqrt{\lambda} L = n\pi, \quad n = 1, 2, 3, \dots$$

Which we need for any nontrivial soln

$$\Rightarrow \sqrt{\lambda_n} = \frac{n\pi}{L}, \quad n = 1, 2, 3, \dots$$

and we have ∞-many soln, with ∞-many different angular freq.

Don't be shocked by this. There were ∞-many problems for each λ.

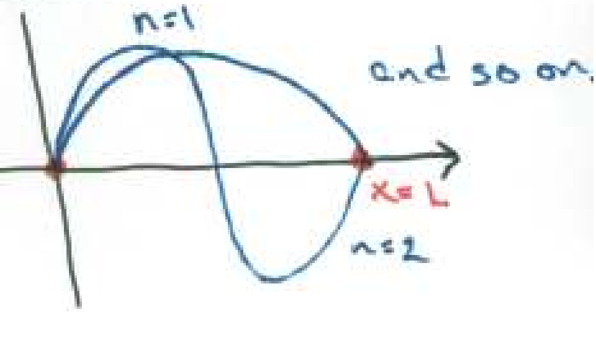
$$y_n(x) = C_n \sin(\sqrt{\lambda_n}\, x), \quad \sqrt{\lambda_n} = \frac{n\pi}{L}, \quad n = 1, 2, 3, \dots$$

It turns out that C_n does not matter.

Notes:

- y_n is called an Eigenfn
- λ_n is called an Eigenvalue

Graphically:



We could have used algebra to show the same thing:

Case $\lambda > 0$:

$$y_1(0) = C_1 \sin(0) + C_2 \cos(0) = \underline{\underline{C_2 = 0}}$$

$$y_1(L) = C_1 \sin(\sqrt{\lambda} L) + \cancelto{0}{C_2} \cos(\sqrt{\lambda} L) = 0$$

$$\Rightarrow \quad \sqrt{\lambda_n} = \frac{n\pi}{L}, \; n = 1, 2, 3, \ldots$$

as before

Case $\lambda < 0$:

$$y_2(0) = C_3 \sinh(0) + C_4 \cosh(0) = \underline{\underline{C_4 = 0}}$$

$$y_2(L) = C_3 \sinh(\sqrt{|\lambda|} L) + \cancelto{0}{C_4} \cosh(\sqrt{|\lambda|} L) =$$

$$= C_3 \underbrace{\left(\frac{e^{\sqrt{|\lambda|} L} - e^{-\sqrt{|\lambda|} L}}{2} \right)}_{\neq 0 \text{ for } L \neq 0 \text{ and } \lambda < 0} = 0 \Rightarrow C_3 = 0$$

Case $\lambda = 0$:

$$y_3(0) = C_5 \cdot 0 + C_6 = 0 \Rightarrow C_6 = 0$$

$$y_3(L) = C_5 \cdot L = 0 \Rightarrow C_5 = 0$$

* Thus, we found just as we did w/ geometry.

(**) The rest of the results can be found in the old soln files.