Note Title 12/7/2007

1

$$\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Char. polynomial

$$\text{Det}\begin{bmatrix} \cos\theta-\lambda & \sin\theta \\ \sin\theta & \cos\theta-\lambda \end{bmatrix} = (\cos\theta-\lambda)^2 - \sin^2\theta = 0$$

$$\cos^2\theta - 2\lambda\cos\theta + \lambda^2 - \sin^2\theta = 0$$

$$-\sin^2\theta = -1 + \cos^2\theta$$

$$(2\cos^2\theta - 1) - 2\lambda\cos\theta + \lambda^2 = 0$$

$$\lambda = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 8\cos^2\theta + 4}}{2}$$

$$4 - 4\cos^2\theta$$
$$= 4(1-\cos^2\theta)$$
$$= 4\sin^2\theta$$

$$\boxed{\lambda = \cos\theta \pm \sin\theta}$$

$\lambda_1 = \cos\theta + \sin\theta$

$\lambda_2 = \cos\theta - \sin\theta$

$$\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix} = (\cos\theta + \sin\theta) \begin{bmatrix} x \\ y \end{bmatrix}}_{\text{1st } \mathcal{E}\text{-vector}}$$

obviously $x = y = 1$ works

$$\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix} = (\cos\theta - \sin\theta) \begin{bmatrix} x \\ y \end{bmatrix}}_{\text{second } \mathcal{E}\text{-vector}}$$

$x = 1 \quad y = -1$

$$\cos\theta + \sin\theta \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \cos\theta - \sin\theta \quad \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

These are the 2 $\mathcal{E}$-value, $\mathcal{E}$-vector pairs.

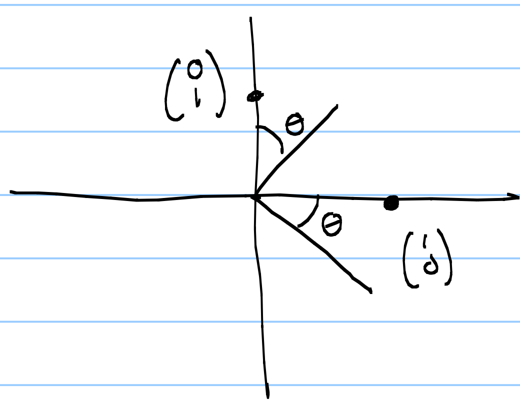
② Let $R_\theta = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

show that R is a rotation matrix.

Action of R_θ on basis vectors.

$$R_\theta \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{pmatrix} \cos\theta \\ -\sin\theta \end{pmatrix}$$

$$R_\theta \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \sin\theta \\ \cos\theta \end{bmatrix}$$



notice that

$$\begin{bmatrix} \cos\theta \\ -\sin\theta \end{bmatrix}^T \begin{bmatrix} \sin\theta \\ \cos\theta \end{bmatrix} = 0$$

what is R_θ for infinitesimal θ

$$R_\theta = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$R_{d\theta} = \underbrace{\begin{bmatrix} 1 & d\theta \\ -d\theta & 1 \end{bmatrix}}$$

E-values E-vectors ?

$$(1-\lambda)(1-\lambda) + d\theta^2 = 0$$

$$1 - 2\lambda + \lambda^2 + 0 = 0$$

$$\lambda = \frac{2 \pm \sqrt{4-4}}{2} \Rightarrow \lambda = 1, 1$$

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(3) Suppose we have $e_1(x), e_2(x)$ .... $e_N(x)$.... be an orthonormal set of functions on $[a,b]$, i.e.,

$$(e_i(x), e_j(x)) \equiv \int_a^b e_i(x)\, e_j(x)\, dx = \delta_{ij}$$

- what does it mean for this set of functions to be complete?

- Derive an expression for the coefficients c_N in

$$f(x) = \sum_{N=0}^{\infty} c_N\, e_N(x) \quad 🚩$$

completeness means that any "reasonable" function on $[a,b]$ can be expressed as in 🚩.

to get the c coefficients I project an arbitrary basis function onto f:

$$(e_m(x), f(x))$$

since $f(x) = \sum_{\ell=0}^{\infty} c_\ell e_\ell(x) \Rightarrow$

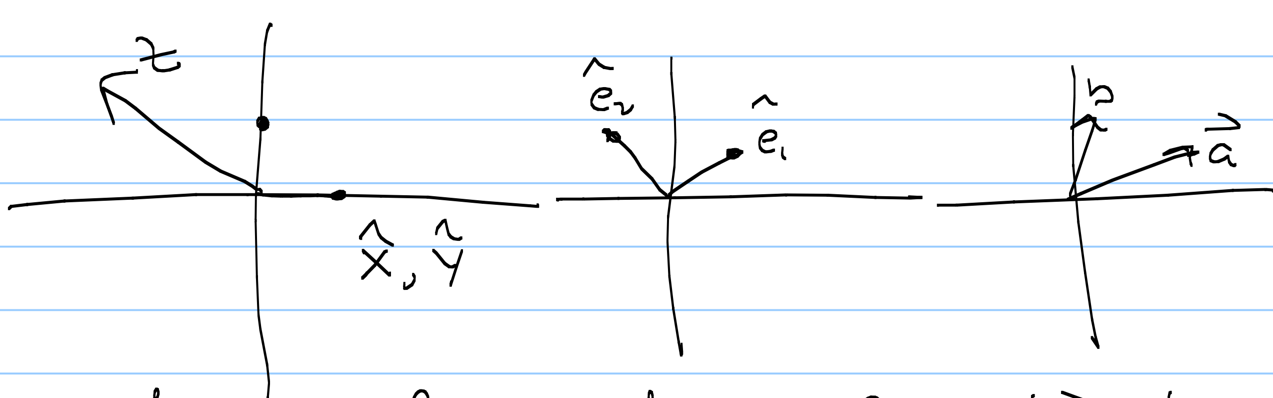
$$(e_m(x), f(x)) = \sum c_\ell \underbrace{(e_m(x), e_\ell(x))}_{\delta_{m\ell}}$$

$$= \sum c_\ell \delta_{m\ell} = c_m$$

so $\quad c_m = (e_m(x), f(x))$

which, by the way, $\displaystyle = \int_a^b e_m(x) f(x)\, dx$

Recall basis vectors



orthonormal $\qquad$ orthonormal $\qquad$ linearly ind.

Key ideas

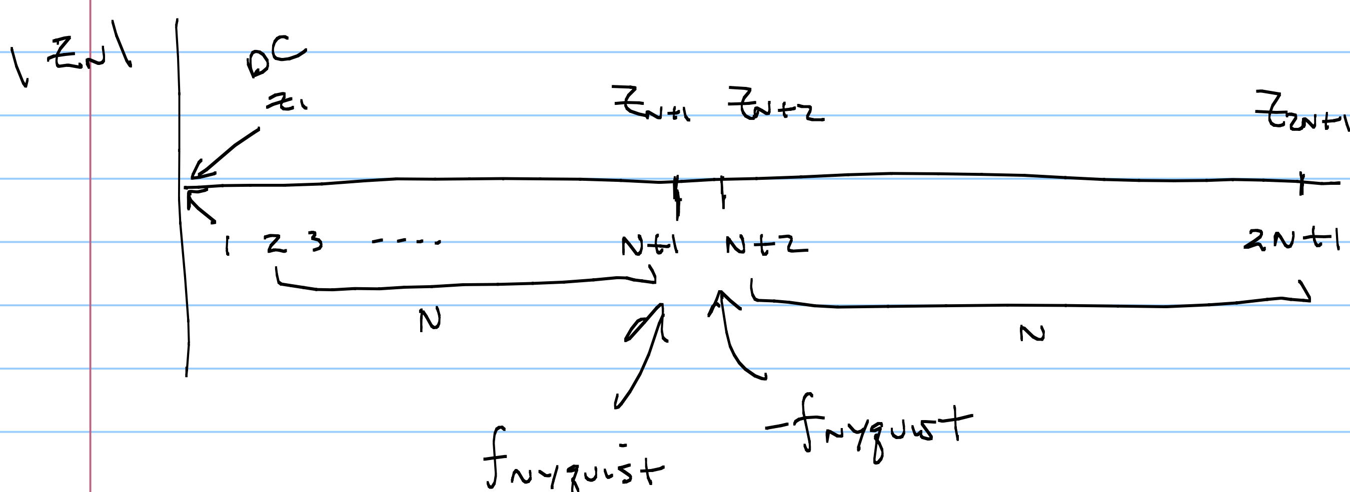
$$\bar{z} = z_x \hat{x} + z_y \hat{y}$$
$$= z_1 \hat{e}_1 + z_2 \hat{e}_2 \quad \Big\} \text{ easy to compute coef}$$

$$= z_a \vec{a} + z_b \vec{b} \quad \Big\} \text{ not so easy but still do-able}$$

$$f(x) = \sum_{\ell=1}^{\infty} A_\ell P_\ell(x)$$
$$\sum_{\ell=-\infty}^{\infty} c_\ell e^{i n \pi x/\ell}$$

Nyquist $\qquad \vec{x} = (x_1, x_2, x_3, \ldots x_{2N+1})$

$$FFT(\vec{x}) \equiv \vec{z} = (z_1, z_2, z_3, \ldots z_{2N+1})$$



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In[4]:= x = Table[Random[], {i, 101}];
ListPlot[Abs[Fourier[x]], Joined → True]
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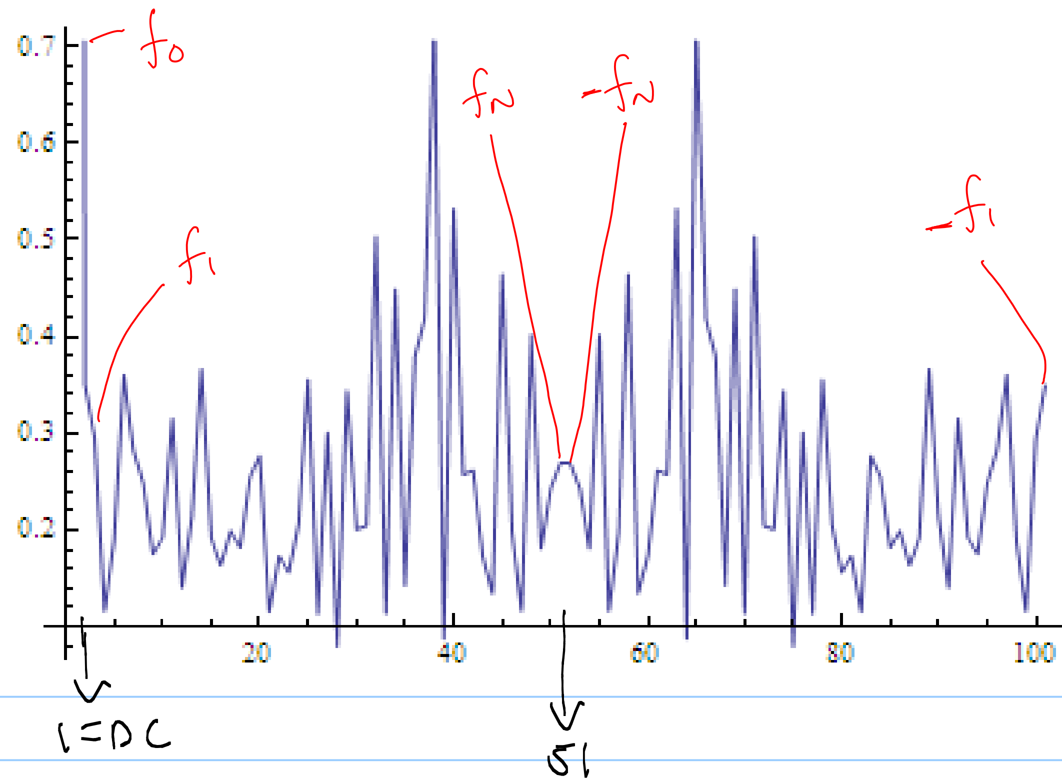
Out[5]=



Diagonalize

4 $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

$$\text{Det} \begin{bmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{bmatrix} = 0 \Rightarrow (1-\lambda)^2 - 4 = 0$$

$$\Rightarrow \lambda^2 - 2\lambda + 1 - 4 = 0$$

$$\lambda_{1,2} = \frac{2 \pm \sqrt{4+12}}{2} = 1 \pm 2$$

$$\lambda_1 = 3 \qquad \lambda_2 = -1$$

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 3 \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{aligned} x + 2y &= 3x \\ 2x + y &= 3y \end{aligned} \qquad x = y \qquad \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = -1 \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{aligned} x + 2y &= -x \\ 2x + y &= -y \end{aligned} \qquad x = -y \qquad \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\vec{q}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \vec{q}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$Q = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$\boxed{Q^T \begin{bmatrix} 3 & \\ & -1 \end{bmatrix} Q = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}}$$

diagonalization