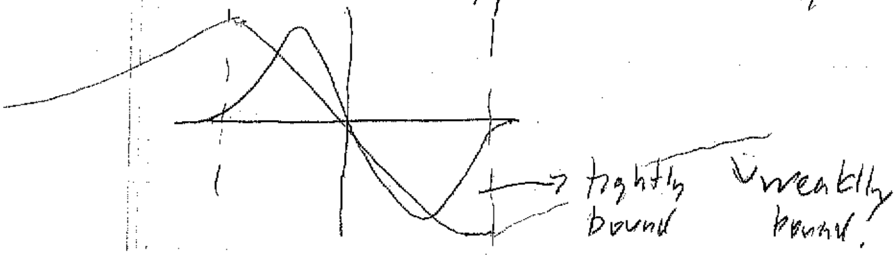
Cutoff condition

As a mode approaches cutoff, more field energy in cladding.



At cutoff, no decay in cladding : $\beta \to 0$

$\beta^2 = k_z^2 - n_2^2 k_0^2 = 0 \quad \to \quad k_z^2 = n_2^2 k_0^2$

this is same conclusion (same physics) as TIR cutoff:

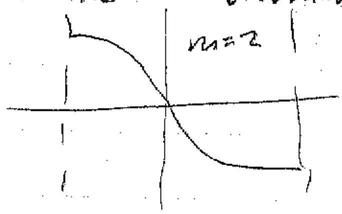
at critical angle, $\quad k_z = k_0 n_1 \sin\theta_c = n_2 k_0$

$\alpha^2 = n_1^2 k_0^2 - k_z^2 \Rightarrow (n_1^2 - n_2^2) k_0^2$ at cutoff

$= V^2/a^2$

since α is transverse wavenumber,

$\alpha = 2\pi/\lambda_T$



at cutoff

$m\frac{\lambda_T}{2} = 2a = \frac{m}{2}\frac{2\pi}{\alpha}$

\# modes

$m_{max} = \frac{2a\alpha}{\pi} = \frac{2}{\pi}V$

take integer value, but m_{max} is always at least 1

single mode if $\quad \frac{2V}{\pi} < 2 \quad$ or $\quad \boxed{\frac{V}{\pi} < 1}$

Cylindrical waveguides

$$\nabla_T^2 E_T - k_z^2 E_T + n_1^2 k_0^2 E_T = 0$$

$\rightarrow$ cyl. coords

$$\frac{1}{r}\partial_r\left(r\,\partial_r E_T\right) + \frac{1}{r^2}\partial_\phi^2 E_T + \left(n_1^2 k_0^2 - k_z^2\right) E_T = 0$$

separation of variables: $E_T(r,\phi) = R(r)\,\Phi(\phi)$

$\rightarrow \frac{1}{\Phi}\partial_\phi^2 \Phi = \text{const.}$ (mult. eqn by $\frac{r^2}{R(\,)\Phi(\,)}$)

$\Phi(\phi) = e^{im\phi}$ $m = \pm$ integer

= azimuthal mode index

expand radial derivatives out, radial eqn is

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + \left(n_1^2 k_0^2 - k_z^2 - \frac{m^2}{r^2}\right) R = 0$$

let $u = k_r r$ $\rightarrow \frac{dR}{dr} = \frac{du}{dr}\frac{dR}{du} = k_r \frac{dR}{du}$

$$\rightarrow k_r^2 \frac{d^2R}{du^2} + \frac{k_r^2}{u}\frac{dR}{du} + \left(n_1^2 k_0^2 - k_z^2 - \frac{k_r^2 m^2}{u^2}\right) R = 0$$

$$u^2 \frac{d^2R}{du^2} + u\frac{dR}{du} + \left[\underbrace{\left(\frac{n_1^2 k_0^2 - k_z^2}{k_r^2}\right)}_{=1} u^2 - m^2\right] R = 0$$

$\rightarrow$ Bessel equation. $J_m(u)$ regular at origin

$N_m(u)$ diverge at origin

both oscillatory.

modified Bessels $I_m(k_r r) = J_m(ik_r r)$ exp growing

$K_m(k_r r) = N_m(ik_r r)$ exp damped.

this is an eigenvalue equation

$$\hat{H}\psi = E\psi \qquad \hat{H} = \frac{\vec{p}^2}{2m} + V(\vec{r})$$

$$= -\frac{\hbar^2}{2m}\nabla^2 + V(\vec{r})$$

how to solve?

correspondence $-\frac{2m}{\hbar^2}V(\vec{r}) \approx \varepsilon(\vec{r})\mu^2$

$-\frac{2mE}{\hbar^2} \approx k_z^2$

- piecewise pot'ls
  - solution to wave eqn. in each region
  - boundary conditions to match E, H
  - → "quantization condition"

limit no z-propagation → resonator w/ mirror ends.

$$\frac{\partial^2 \vec{E}}{\partial x^2} + n_0^2 k_{vac}^2 \vec{E} = 0$$

$x=0$ ⟷ L

$$\vec{E} = E_1 e^{ikx} + E_2 e^{-ikx} \qquad k = n_1 k_{vac}$$

apply boundary cond.

$x=0 \qquad E_1 + E_2 = 0 \qquad \to E = E_0 \sin kx$

$X = L \qquad \sin kL = 0 \qquad k_m L = m\pi \qquad m = 1, 2, 3 \ldots$

allowed $k_m = \frac{m\pi}{L} \qquad m=0 \to \lambda = L/2$

m = mode index $\qquad = \underline{\text{cutoff}}$

allowed freq: $\omega_m = k_m c/n_0$ spacing: $\frac{\pi c}{n_0 L} = \frac{2\pi}{T_{\text{round trip}}}$

note both polarizations are allowed.

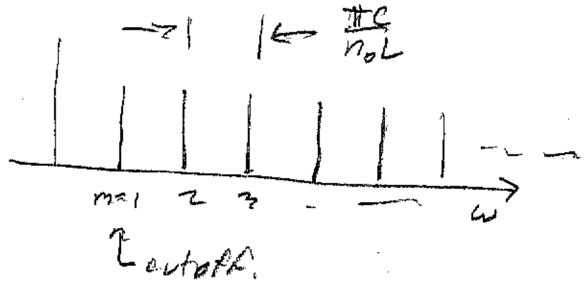
e.g. $\vec{E} = \hat{y}E_0 \underbrace{\sin k_m x}_{\text{standing wave in } x} e^{-i\omega t}$

$\vec{B}$? $\qquad k_m E_0 \cos k_m x = -\frac{i\omega}{c} B_0(x)$

B is 90° out of phase.

In a resonator, only discrete frequencies are allowed.

spectrum



dispersion relation

$$n_o^2 k_v^2 = k_m^2 \qquad \text{only } x\text{-component.}$$

$$\omega^2 = k_m^2 c^2 / n_o^2$$

waveguide : allow for unconfined propagation in z

$$\frac{\partial^2 \vec{E}}{\partial z^2} + \frac{\partial^2 E}{\partial x^2} + n_o^2 k_{vac}^2 \vec{E} = 0$$

soln's are still plane waves: $e^{\pm i k_x x}$, $e^{\pm i k_z z}$

$$\rightarrow \quad -k_z^2 - k_x^2 + n_o^2 k_{vac}^2 = 0$$

vector relationship btw. components comes directly from wave eqn.

$$k_{xm} = \frac{m\pi}{L} \quad \text{still}$$

$$n_o^2 k_{vac}^2 = \left(\frac{m\pi}{L}\right)^2 + k_z^2$$

or
$$\omega^2 = \frac{c^2}{n_o^2}\left[\left(\frac{m\pi}{L}\right)^2 + k_z^2\right]$$

k_z free to vary, allowed ω is continuous now

k_z = real → propagating waves.

$k_z = 0 \quad \omega = \omega_{cutoff} = \frac{c\pi}{n_o L} m$

"single-mode"

Each mode has a cutoff – range where only m=1