PHGN 462 Homework 3

1) (From Pollack and Stump 6.17)

Consider an electric field line passing through a planar interface between two insulating media with dielectric constants κ_1 and κ_2. Assume there is no free charge on the interface. Let θ_1 and θ_2 be the angles between the field line and the normal to the interface in the two regions. Prove that
$\kappa_1 \cot\theta_1 = \kappa_2 \cot\theta_2$

Note that this problem is basically about applying our boundary conditions in matter. As part of your answer, include a diagram of the electric field lines in the two regions and the angles they make. Specify which region has the larger κ, and make your diagram consistent with that. The form of the equation we're proving and the diagram you're making should be very reminiscent of Snell's law. Coincidence? We'll find out when we do electromagnetic waves and derive Snell's law.

2) (From Pollack and Stump 11.34) This problem gives us a look at the actual mechanical effects of fields on particles, and shows us how to treat the general problem numerically (real problems almost always have numerical solutions). One thing I like about it is that it shows how much you can solve for even in a very general case – for most of the problem, we never specify the potential exactly, but still manage to do quite a bit of work.

Consider an electromagnetic wave with vector potential $\vec{A}(\vec{x},t) = \hat{\jmath} f(x - ct)$. (The scalar potential is 0.) The function $f(x - ct)$ approaches 0 as $x \to \pm\infty$, so the electromagnetic field is a wave packet (that is, a wave localized in space). Suppose the wave hits an electron (charge –e, mass m) initially at rest at the origin.

a) Derive the equations of motion for the electron velocity components v_x, v_y, v_z.

b) Show that $v_z(t) = 0$.

c) Show that $v_y(t) = \left(\frac{e}{m}\right) f(x - ct)$, where x is the x position of the electron at time t.

d) Show that $v_x\left(1 - \frac{v_x}{2c}\right) = \frac{e^2}{2m^2c}[f(x - ct)]^2$

e) Describe the trajectory of the electron, assuming the wave packet has a short length. In particular, show that the electron will have a positive displacement in the x direction.

f) Assume $v_x \ll c$ and $f(\xi) = Ke^{-\xi^2/d^2}$. Use a computer to solve the differential equation for x(t) numerically and plot the result. (Let d be the unit of length, d/c the unit of time, and choose a relatively small value of the dimensionless constant $eK/(mc)$.)

g) Solve for y(t) and plot the trajectory in space. For example, in Mathematica, solve simultaneously the coupled parametric equations for x(t) and y(t) using NDSolve and make the plot with ParametricPlot.

Some clarifications/hints follow:

For c & d you have to be super careful with the difference between total and partial derivatives, and attend to the fact that f is a function of $(x - ct)$. It's tricky. Fortunately, it's a "show that" problem, so if you get stuck on c & d you can still do the rest. I found it useful to define $\eta = x - ct$ so as to write $f(x - ct)$ as $f(\eta)$. That helped me keep track of all the multivariate chain rule stuff. Though I still had to write things out in almost embarrassing detail to get all the terms to show up right.

For f, it's implied that $\xi = x - ct$. Also, "relatively small" is kind of vague. Let's say "relatively small" is 0.1.

For f also note that if they say, for example, that d is a "unit" of length, that means d has unit value. Which is to say, it's 1. And if d is unit and d/c is unit, so must be c. Really, we're just trying to get rid of all the constants so we can zoom in on the qualitative behavior of the system.