1_28_08

Note Title 1/28/2008

Quantum Harmonic Oscillator

$$\frac{p^2}{2m} + \frac{1}{2}kx^2 \Rightarrow \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2$$

where $p = -i\hbar \frac{d}{dx}$

Idea is to factor H (it's quadratic in p & x)

Define $a = \sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{i}{m\omega}p\right)$

$$a^+ = \sqrt{\frac{m\omega}{2\hbar}}\left(x - \frac{i}{m\omega}p\right)$$

obviously a is not Hermitian

NB

$$a + a^+ = \sqrt{\frac{2m\omega}{\hbar}}\, x$$

so $\boxed{x = \sqrt{\frac{\hbar}{2m\omega}}\left(a + a^+\right)}$

$$a^+ - a = -i\sqrt{\frac{2}{m\omega\hbar}}\, p$$

$\Rightarrow$ $\boxed{p = i\sqrt{\frac{m\omega\hbar}{2}}\left(a^+ - a\right)}$

$$a^{+}a = \frac{m\omega}{2\hbar}\left(x - \frac{i}{m\omega}p\right)\left(x + \frac{i}{m\omega}p\right)$$

$$= \frac{m\omega}{2\hbar}\left(x^2 \underbrace{- \frac{i}{m\omega}px + \frac{i}{m\omega}xp} + \frac{1}{m^2\omega^2}p^2\right)$$

cross terms don't cancel since p and x don't commute

$$[x,p] = xp - px = i\hbar$$

$$a^{+}a = \frac{m\omega}{2\hbar}\left(x^2 + \frac{i}{m\omega}[x,p] + \frac{1}{m^2\omega^2}p^2\right)$$

$$= \frac{m\omega}{2\hbar}\left(x^2 - \frac{\hbar}{m\omega} + \frac{1}{m^2\omega^2}p^2\right)$$ careful

$$= \frac{1}{\hbar\omega}\frac{m\omega^2}{2}\left(x^2 + \frac{1}{m^2\omega^2}p - \frac{\hbar}{m\omega}\right)$$

$$= \frac{1}{\hbar\omega}\left[\underbrace{\tfrac{1}{2}m\omega^2x^2 + \tfrac{1}{2m}p^2}_{H} - \tfrac{1}{2}\hbar\omega\right]$$

$$a^{+}a = \frac{1}{\hbar\omega}\left[H - \tfrac{1}{2}\hbar\omega\right]$$

$$= \frac{1}{\hbar\omega}H - \frac{1}{2}$$

$$\boxed{H = \hbar\omega\left(a^{+}a + \tfrac{1}{2}\right)}$$

So a and a^+ allow us to factor H with a little bit left over.

$\frac{1}{2}\hbar\omega$ will turn out to be the QM ground stat energy

we need the following results

1) $[a, a^+] = 1$

2) $[H, a] = -\hbar\omega a$

3 $[H, a^+] = \hbar\omega a^+$

we'll prove 1) in class. do the other 2 for practice

$$[a, a^+] = a a^+ - a^+ a$$

$$= \frac{m\omega}{2\hbar}\left(x + \frac{i}{m\omega}p\right)\left(x - \frac{i}{m\omega}p\right)$$

$$- \frac{m\omega}{2\hbar}\left(x - \frac{i}{m\omega}p\right)\left(x + {}^{i}/_{m\omega}\, p\right)$$

$$= \frac{m\omega}{2\hbar}\left\{\left[x^2 \underbrace{- \frac{i}{m\omega} xp + \frac{i}{m\omega} px}_{-\frac{i}{m\omega}[x,p]} + \frac{1}{m^2\omega^2} p^2\right]\right.$$

$$\left. - \left[x^2 \underbrace{+ \frac{i}{m\omega} xp - \frac{i}{m\omega} px}_{+\frac{i}{m\omega}[x,p]} + \frac{1}{m^2\omega^2} p^2\right]\right\}$$

$$\frac{m\omega}{2\hbar}\left\{\left[x^2 + \frac{\hbar}{m\omega} + \frac{1}{m^2\omega^2} p^2\right] - \left[x^2 - \frac{\hbar^2}{m\omega} + \frac{1}{m^2\omega^2} p^2\right]\right\}$$

$$\frac{m\omega}{2\hbar} \quad \frac{2\hbar}{m\omega} = 1$$

$$[a, a^+] = 1$$

QED

The other 2 formulae

2) $[H, a] = -\hbar\omega a$

3 $[H, a^+] = \hbar\omega a^+$

will let us prove the following deep results.

if $H\psi_E = E\psi_E$

The $a\psi_E$ is also an $\mathcal{E}$-vector with $\mathcal{E}$-value $E - \hbar\omega$.

$a^+\psi_E$ is an $\mathcal{E}$-vector with $\mathcal{E}$-value $E + \hbar\omega$

The importance of this is: all we need is one $\mathcal{E}$-vector and we can generate all the rest algebraically

$a\psi_E$

since $[H, a] = Ha - aH = -\hbar\omega a$

$$H a\psi_E = aH\psi_E - \hbar\omega a\psi_E$$

$$= aE\psi_E - \hbar\omega a\psi_E$$

$$= (E - \hbar\omega) a\psi_E$$

so $$H \underbrace{a\psi_E}_{\mathcal{E}\text{-vector}} = \underbrace{(E - \hbar\omega)}_{\mathcal{E}\text{-value}} \underbrace{a\psi_E}_{\mathcal{E}\text{-vector}}$$

$$a^+ \psi_E$$

$$[H, a^+] = Ha^+ - a^+H = \hbar\omega a^+$$

$$Ha^+\psi_E = a^+ H \psi_E + \hbar\omega\, a^+ \psi_E$$

$$= E\, a^+ \psi_E + \hbar\omega\, a^+ \psi_E$$

$$\underline{H\, a^+ \psi_E} = (E + \hbar\omega)\, a^+ \psi_E$$

now any vector satisfies

$(x, x) \geq 0$ so

$$(a\psi_E, a\psi_E) \geq 0$$

$$= (\psi_E, a^+ a \psi_E)$$

we prooved $a^+ a = \frac{1}{\hbar\omega} H - \frac{1}{2}$

So

$$(\psi_E, a^+ a \psi_E) = (\psi_E, \tfrac{1}{\hbar\omega} H - \tfrac{1}{2} \psi_E)$$

$$= (\psi_E, \tfrac{1}{\hbar\omega} H \psi_E) + (\psi_E, -\tfrac{1}{2} \psi_E)$$

$$= \frac{1}{\hbar\omega}(\psi_E, E\psi_E) - \frac{1}{2}(\psi_E, \psi_E)$$

$$= \frac{E}{\hbar\omega}\underbrace{(\psi_E, \psi_E)}_{1} - \frac{1}{2}\underbrace{(\psi_E, \psi_E)}_{1}$$

so $\quad \dfrac{E}{\hbar\omega} - \dfrac{1}{2} \geq 0$

$$\Rightarrow \quad \boxed{E \geq \frac{\hbar\omega}{2}}$$

$\dfrac{\hbar\omega}{2}$ is the "Ground state" energy.

the stationary state associated with this is defined to be ψ_0

$$H\psi_0 = \frac{\hbar\omega}{2}\psi_0$$

we've shown that

$$H \underbrace{a^+\psi_0}_{\text{call this } \psi_1} = \left(\hbar\omega + \frac{\hbar\omega}{2}\right)\underbrace{a^+\psi_0}_{\psi_1}$$

$$H \psi_1 = \left(\hbar\omega + \frac{\hbar\omega}{2}\right) \psi_1$$

$$H \underbrace{a^+ \psi_1}_{\psi_2} = \left(2\hbar\omega + \frac{\hbar\omega}{2}\right) \underbrace{a^+ \psi_1}_{\psi_2}$$

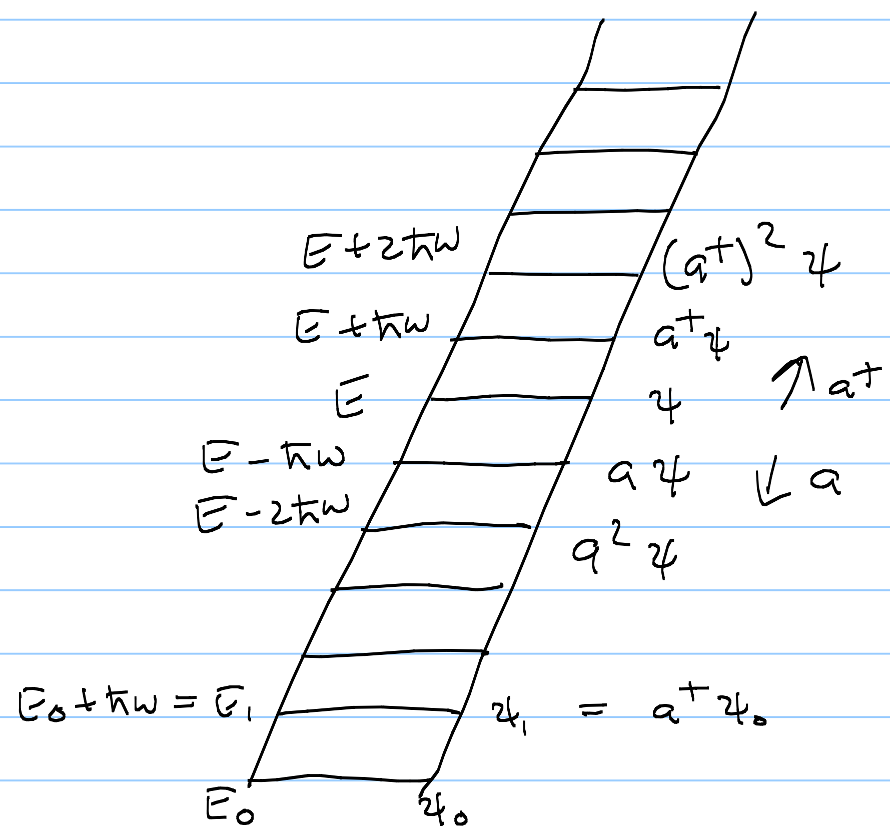
$$\vdots$$

$$H \psi_N = \left(N\hbar\omega + \frac{\hbar\omega}{2}\right) \psi_N$$

$$\boxed{H \psi_N = \hbar\omega \left(N + \frac{1}{2}\right) \psi_N}$$

The QHO has a ladder of equally spaced energy levels

$$\boxed{E_N - E_{N-1} = \hbar\omega}$$



a and a^+ are called:

ladder operators

or

creation (a^+) annihilation (a) operators.

But how do we determine ψ_0? since its the ground state

$$a\psi_0 = 0$$

$$\sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{i}{m\omega}p\right)\psi_0 = 0$$

$$x\psi_0 + \frac{\hbar}{m\omega}\frac{d\psi_0}{dx} = 0$$

$$\boxed{\frac{d\psi_0}{dx} + \frac{m\omega}{\hbar}x\,\psi_0 = 0}$$

<u>guess</u> $\quad e^{-\alpha x^2}$

$$\frac{d}{dx} = -2\alpha x\, e^{-\alpha x^2}$$

so $\quad \boxed{\psi_0 = A e^{-\frac{m\omega}{2\hbar}x^2}}$

Normalization

$$\int_{-\infty}^{\infty}|\psi_0|^2 dx = A^2\int_{-\infty}^{\infty} e^{-\frac{m\omega}{\hbar}x^2}\,dx = 1$$

$$\Rightarrow A^2 = \sqrt{\frac{m\omega}{\pi\hbar}}$$

$$\boxed{\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2}}$$

we now have everything but the normalization

$$\psi_n(x) = A_n \, (a^+)^n \, \psi_0$$

$\hookrightarrow$ normalization

$$E_n = \hbar\omega \, (n + 1/2)$$

Turns out (hard calculation)

$$A_n = \frac{1}{\sqrt{n!}}$$

$$\boxed{\psi_n(x) = \frac{1}{\sqrt{n!}} \, (a^+)^n \, \psi_0}$$