

Quote of Homework Five Solutions

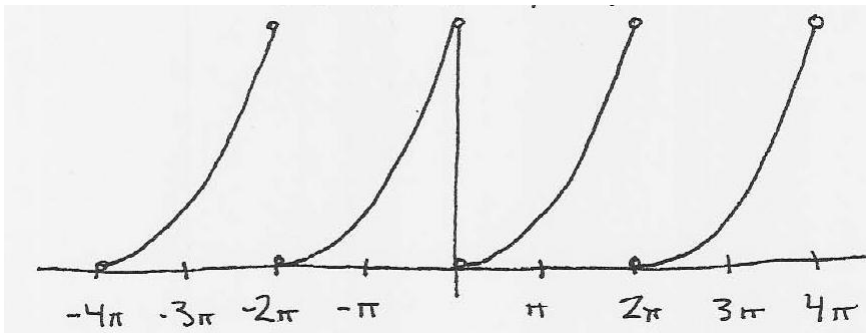
And you know you're never sure, but you're sure you could be right if you held yourself up to the light.

The Smashing Pumpkins : Mellon Collie and Infinite Sadness (1995)

1. FOURIER SERIES : NONSTANDARD DOMAIN

Let $f(x) = x^2$ for $x \in (0, 2\pi)$ be such that $f(x + 2\pi) = f(x)$.

1.1. **Graphing.** Sketch f on $(-4\pi, 4\pi)$.



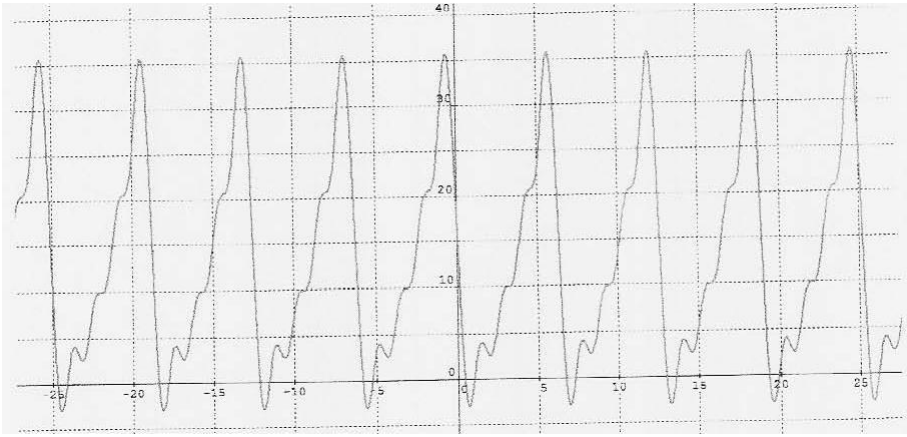
1.2. **Symmetry.** Is the function even, odd or neither?

Don't let the quadratic function fool you. This function is neither even nor odd as can be seen by the previous graph.

1.3. **Integrations.** Determine the Fourier coefficients a_0, a_n, b_n of f .

$$\begin{aligned}
 a_0 &= \frac{1}{2\pi} \int_0^{2\pi} x^2 dx = \frac{x^3}{6\pi} \Big|_0^{2\pi} = \frac{4\pi^2}{3} \\
 a_n &= \frac{1}{\pi} \int_0^{2\pi} x^2 \cos(nx) dx = \\
 &= \frac{1}{\pi} \left[\frac{x^2}{n} \sin(nx) + \frac{2x}{n^2} \cos(nx) - \frac{2}{n^3} \sin(nx) \right]_0^{2\pi} \\
 &= \frac{1}{\pi} \left[\frac{4\pi^2}{n^2} \right] = \frac{4}{n^2} \\
 b_n &= \frac{1}{\pi} \int_0^{2\pi} x^2 \sin(nx) dx \\
 &= \frac{1}{\pi} \left[\frac{-x^2}{n} \cos(nx) + \frac{2x}{n^2} \sin(nx) + \frac{2}{n^3} \cos(nx) \right]_0^{2\pi} \\
 &= \frac{1}{\pi} \left[\frac{-4\pi^2}{n} + \frac{2}{n^3} - \frac{2}{n^3} \right] = \frac{-4\pi}{n} \\
 f(x) &= \frac{4\pi^2}{3} + \sum_{n=1}^{\infty} \left[\frac{4}{n^2} \cos(nx) - \frac{4\pi}{n} \sin(nx) \right] \\
 a_0 &= \frac{4\pi^2}{3} \quad a_n = \frac{4}{n^2} \quad b_n = \frac{-4\pi}{n}
 \end{aligned}$$

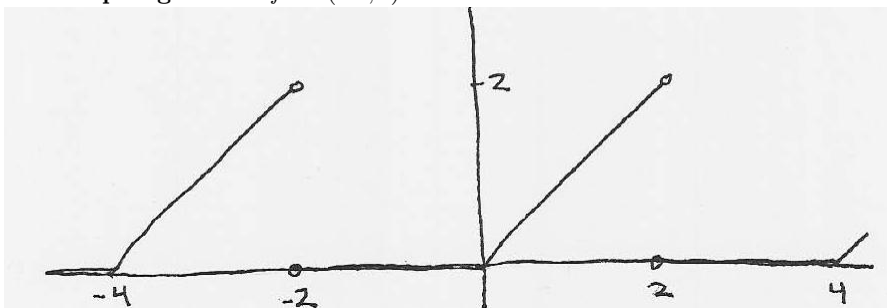
1.4. **Truncation.** Using <http://www.tutor-homework.com/grapher.html>, or any other graphing tool, graph the first five terms of your Fourier Series Representation of f .



2. FOURIER SERIES : NONSTANDARD PERIOD

Let $f(x) = \begin{cases} 0, & -2 < x < 0 \\ x, & 0 < x < 2 \end{cases}$ be such that $f(x+4) = f(x)$.

2.1. **Graphing.** Sketch f on $(-4, 4)$.



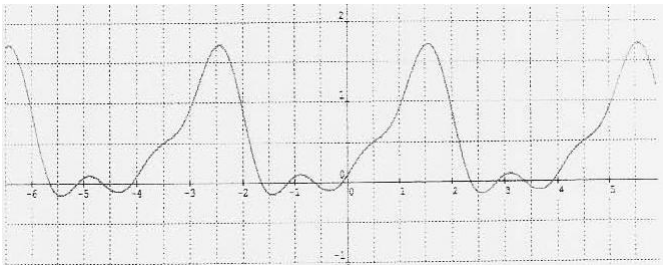
2.2. **Symmetry.** Is the function even, odd or neither?

This function is neither even nor odd.

2.3. **Integrations.** Determine the Fourier coefficients a_0, a_n, b_n of f .

$$\begin{aligned}
a_0 &= \frac{1}{2L} \int_{-1}^1 f(x) dx = \frac{1}{4} \left[\int_{-2}^0 0 dx + \int_0^2 x dx \right] = \\
&= \left. \frac{1}{8} x^2 \right|_0^2 = \frac{1}{2} \\
a_n &= \frac{1}{L} \int_{-L}^L f(x) \cos\left(\frac{n\pi}{L}x\right) dx = \\
&= \frac{1}{2} \left[\int_{-2}^0 0 \cdot \cos\left(\frac{n\pi}{2}x\right) dx + \int_0^2 x \cdot \cos\left(\frac{n\pi}{2}x\right) dx \right] = \\
&= \frac{1}{2} \left[\frac{2x}{n\pi} \sin(n\pi x) + \frac{4}{n^2\pi^2} \cos\left(\frac{n\pi}{2}x\right) \right]_0^2 = \\
&= \frac{1}{2} \left[\frac{4}{n^2\pi^2} \cos(n\pi) - \frac{4}{n^2\pi^2} \right] = \frac{2(-1)^n - 2}{n^2\pi^2} \\
b_n &= \frac{1}{L} \int_{-L}^L f(x) \sin\left(\frac{n\pi}{L}x\right) dx = \\
&= \frac{1}{2} \left[\int_{-2}^0 0 \cdot \sin\left(\frac{n\pi}{2}x\right) + \int_0^2 x \cdot \sin\left(\frac{n\pi}{2}x\right) dx \right] = \\
&= \frac{1}{2} \left[\frac{-2x}{n\pi} \cos\left(\frac{n\pi}{2}x\right) + \frac{4}{n^2\pi^2} \sin\left(\frac{n\pi}{2}x\right) \right]_0^2 = \\
&= \frac{1}{2} \left[\frac{-4}{n\pi} (-1)^n \right] = \frac{2(-1)^{n+1}}{n\pi} \\
f(x) &= \frac{1}{2} + \sum_{n=1}^{\infty} \left[\frac{2(-1)^n - 2}{n^2\pi^2} \cos\left(\frac{n\pi}{2}x\right) + \frac{2(-1)^{n+1}}{n\pi} \sin\left(\frac{n\pi}{2}x\right) \right] \\
a_0 &= \frac{1}{2} \quad a_n = \frac{2(-1)^n - 2}{n^2\pi^2} \quad b_n = \frac{2(-1)^{n+1}}{n\pi}
\end{aligned}$$

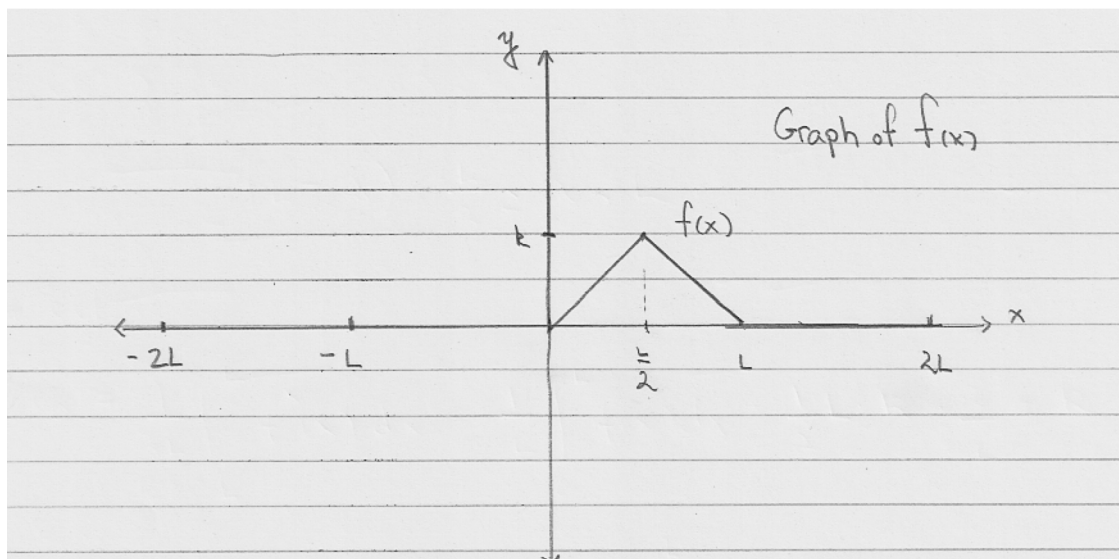
2.4. Truncation. Using <http://www.tutor-homework.com/grapher.html>, or any other graphing tool, graph the first five terms of your Fourier Series Representation of f .



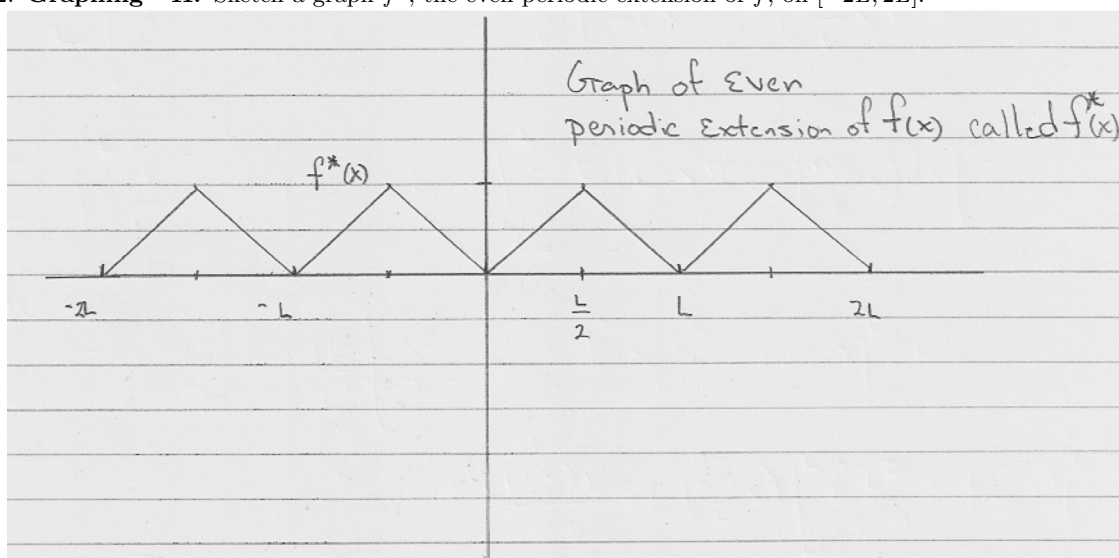
3. FOURIER SERIES : PERIODIC EXTENSION

$$\text{Let } f(x) = \begin{cases} \frac{2k}{L}x, & 0 < x \leq \frac{L}{2} \\ \frac{2k}{L}(L-x), & \frac{L}{2} < x < L \end{cases}.$$

3.1. Graphing - I. Sketch a graph f on $[-2L, 2L]$.



3.2. **Graphing - II.** Sketch a graph f^* , the even periodic extension of f , on $[-2L, 2L]$.



3.3. **Fourier Series.** Calculate the Fourier cosine series for the half-range expansion of f .

We begin with the coefficient a_0 and noting that this is nothing more than the area under the curve $f(x)$ we find,

$$(3.1) \quad a_0 = \frac{1}{2L} \int_{-L}^L f^*(x) dx$$

$$(3.2) \quad = \frac{1}{L} \int_0^L f(x) dx$$

$$(3.3) \quad = \frac{1}{L} \cdot \frac{1}{2} Lk$$

$$(3.4) \quad = \frac{k}{2}$$

Next we have a_n . Some symmetry, integration by parts and algebra gives,

$$(3.5) \quad a_n = \frac{1}{L} \int_{-L}^L f^*(x) \cos\left(\frac{n\pi}{L}x\right) dx$$

$$(3.6) \quad = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi}{L}x\right) dx$$

$$(3.7) \quad = \frac{2}{L} \left[\frac{2k}{L} \int_0^{\frac{L}{2}} x \cos\left(\frac{n\pi}{L}x\right) dx + \frac{2k}{L} \int_{\frac{L}{2}}^L (L-x) \cos\left(\frac{n\pi}{L}x\right) dx \right]$$

$$(3.8) \quad = \frac{4k}{L^2} \left[\frac{L}{n\pi} \sin\left(\frac{n\pi}{L}x\right) \Big|_0^{\frac{L}{2}} + \frac{L^2}{n^2\pi^2} \cos\left(\frac{n\pi}{L}x\right) \Big|_0^{\frac{L}{2}} + \frac{L}{n\pi} (L-x) \sin\left(\frac{n\pi}{L}x\right) \Big|_{\frac{L}{2}}^L - \frac{L^2}{n^2\pi^2} \cos\left(\frac{n\pi}{L}x\right) \Big|_{\frac{L}{2}}^L \right]$$

$$(3.9) \quad = \frac{4k}{L^2} \left[\frac{L^2}{2n\pi} \sin\left(\frac{n\pi}{2}\right) + \frac{L^2}{n^2\pi^2} \cos\left(\frac{n\pi}{2}\right) - \frac{L^2}{n^2\pi^2} - \frac{L^2}{2n\pi} \sin\left(\frac{n\pi}{2}\right) - \frac{L^2}{n^2\pi^2}(-1)^n + \frac{L^2}{n^2\pi^2} \cos\left(\frac{n\pi}{2}\right) \right]$$

$$(3.10) \quad = \frac{4k}{n^2\pi^2} \left[2 \cos\left(\frac{n\pi}{2}\right) - (-1)^n - 1 \right]$$

Further simplifications can be made. If we note the following pattern,

$$(3.11) \quad n = 1 \implies a_1 = 0,$$

$$(3.12) \quad n = 2 \implies a_2 = -\frac{16k}{2^2n^2},$$

$$(3.13) \quad n = 3 \implies a_3 = 0,$$

$$(3.14) \quad n = 4 \implies a_4 = 0,$$

$$(3.15) \quad n = 5 \implies a_5 = 0,$$

$$(3.16) \quad n = 6 \implies a_6 = -\frac{16k}{6^2n^2},$$

we can write the Fourier cosine series as,

$$(3.17) \quad f(x) = \frac{k}{2} + \sum_{n=1}^{\infty} \frac{4k}{n^2\pi^2} \left[2 \cos\left(\frac{n\pi}{2}x\right) - (-1)^n - 1 \right] \cos\left(\frac{n\pi}{L}x\right)$$

$$(3.18) \quad = \frac{k}{2} - \frac{16k}{\pi^2} \left(\frac{1}{2^2} \cos\left(\frac{2\pi}{L}x\right) + \frac{1}{6^2} \cos\left(\frac{6\pi}{L}x\right) + \dots \right)$$

4. COMPLEX FOURIER SERIES

4.1. Orthogonality Results. Show that $\langle e^{inx}, e^{-imx} \rangle = 2\pi\delta_{nm}$ where $n, m \in \mathbb{Z}$, where $\langle f, g \rangle = \int_{-\pi}^{\pi} f(x)g(x)dx$.

$$(4.1) \quad \text{For } n \neq m \quad \int_{-\pi}^{\pi} e^{inx} e^{-imx} dx = \int_{-\pi}^{\pi} e^{(n-m)x} dx = \frac{e^{(n-m)x}}{i(n-m)} \Big|_{-\pi}^{\pi} =$$

$$(4.2) \quad = \frac{(-1)^{(n-m)}}{i(n-m)} - \frac{(-1)^{(n-m)}}{i(n-m)} = 0$$

$$(4.3) \quad \text{For } n = m \quad \int_{-\pi}^{\pi} e^{inx} e^{-imx} dx = \int_{-\pi}^{\pi} e^{(n-m)x} dx = \int_{-\pi}^{\pi} 1 dx = x \Big|_{-\pi}^{\pi} = 2\pi$$

$$(4.4) \quad = \int_{-\pi}^{\pi} e^{inx} e^{-imx} dx = \begin{cases} 0 & n \neq m \\ 2\pi & n = m \end{cases} = 2\pi\delta_{nm}$$

4.2. Fourier Coefficients. Using the previous orthogonality relation find the Fourier coefficients, c_n , for the complex Fourier series,

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}.$$

$$\begin{aligned}
f(x) &= \sum_{n=-\infty}^{\infty} c_n e^{inx} \\
\Rightarrow f(x)e^{-imx} &= \sum_{n=-\infty}^{\infty} c_n e^{inx} e^{-imx} \\
\Rightarrow \int_{-\infty}^{\infty} f(x)e^{-imx} dx &= \sum_{n=-\infty}^{\infty} \int_{-\pi}^{\pi} c_n e^{(n-m)x} dx \\
&\text{As we found in (a), the integral on the right is 0 for all values of } n \text{ except } n=m \\
\Rightarrow \int_{-\pi}^{\pi} f(x)e^{-imx} dx &= c_m 2\pi \\
\Rightarrow \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)e^{-imx} dx &= c_m \\
c_n &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)e^{-inx} dx
\end{aligned}$$

Because $m=n$ we can replace our m 's with n 's to get the formula for c_n

4.3. Complex Fourier Series Representation. Find the complex Fourier coefficients for $f(x) = x^2$, $-\pi < x < \pi$, $f(x+2\pi) = f(x)$.

$$\begin{aligned}
(4.5) \quad f(x) &= \sum_{n=-\infty}^{\infty} c_n e^{inx} \\
(4.6) \quad c_n &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)e^{-inx} dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} x^2 e^{-inx} dx \\
(4.7) &= \frac{1}{2\pi} \left[\frac{-x^2}{in} e^{-inx} + \frac{2x}{n^2} e^{-inx} + \frac{2}{in^3} e^{-inx} \right]_{-\pi}^{\pi} \\
(4.8) &= \frac{1}{2\pi} \left[\left(\frac{-x^2}{in} + \frac{2x}{n^2} + \frac{2}{in^3} \right) e^{-inx} \right]_{-\pi}^{\pi} \\
(4.9) &= \frac{1}{2\pi} \left[\left(\frac{-\pi^2}{in} + \frac{2\pi}{n^2} + \frac{2}{in^3} + \frac{\pi^2}{in} + \frac{2\pi}{n^2} - \frac{2}{in^3} \right) (-1)^n \right] \\
(4.10) &= \frac{1}{2\pi} \left[\frac{4\pi}{n^2} (-1)^n \right] = \frac{2}{n^2} (-1)^n \quad n \neq 0 \\
(4.11) &\text{For } n = 0 \\
(4.12) \quad c_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} x^2 e^{i(0)x} dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} x^2 dx = \frac{1}{2\pi} \left[\frac{1}{3} x^3 \right]_{-\pi}^{\pi} = \\
(4.13) &= \frac{\pi^2}{3} \\
(4.14) \quad f(x) &= \frac{\pi^2}{3} + \sum_{n=-\infty, n \neq 0}^{\infty} \frac{2}{n^2} (-1)^n e^{inx}
\end{aligned}$$

4.4. Conversion to Real Fourier Series. Using the complex Fourier series representation of f recover its associated real Fourier series.

$$(4.15) \quad f(x) = \frac{\pi^2}{3} + \sum_{n=-\infty, n \neq 0}^{\infty} \frac{2}{n^2} (-1)^n e^{inx}$$

$$(4.16) \quad = \frac{\pi^2}{3} + \sum_{n=-\infty}^{-1} \frac{2}{n^2} (-1)^n e^{inx} + \sum_{n=1}^{\infty} \frac{2}{n^2} (-1)^n e^{inx}$$

$$(4.17) \quad \text{Substituting } n=-n \text{ into the first series we get:}$$

$$(4.18) \quad = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{2}{n^2} (-1)^n e^{-inx} + \sum_{n=1}^{\infty} \frac{2}{n^2} (-1)^n e^{inx}$$

$$(4.19) \quad = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{2}{n^2} (-1)^n (e^{-inx} + e^{inx})$$

$$(4.20) \quad \text{Using Euler's Formula:}$$

$$(4.21) \quad = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{2}{n^2} (-1)^n (\cos(nx) - i \sin(nx) + \cos(nx) + \sin(nx))$$

$$(4.22) \quad \text{The Real Fourier Series Representation:}$$

$$(4.23) \quad f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} (-1)^n \cos(nx)$$

5. PERIODIC FORCING OF SIMPLE HARMONIC OSCILLATORS

Consider the ODE, which is commonly used to model forced simple harmonic oscillation,

$$(5.1) \quad y'' + 9y = f(t),$$

$$(5.2) \quad f(t) = |t|, \quad -\pi \leq t < \pi, \quad f(t + 2\pi) = f(t).$$

Since the forcing function (5.2) is a periodic function we can study (5.1) by expressing $f(t)$ as a Fourier series.^{1 2}

5.1. Fourier Series Representation. Express $f(t)$ as a real Fourier series.

If we graph $f(t)$ we should notice that its graph is the same as the graph as the function from problem 3 where $k = \pi$ and $L = 2\pi$. Thus,

$$(5.3) \quad a_0 = \frac{\pi}{2}$$

$$(5.4) \quad a_n = \frac{4}{n^2\pi} \left(2 \cos\left(\frac{n\pi}{2}\right) - (-1)^n - 1 \right),$$

which gives the Fourier series representation of $f(t)$ as,

$$(5.5) \quad f(t) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{4}{n^2\pi} \left(2 \cos\left(\frac{n\pi}{2}\right) - (-1)^n - 1 \right) \cos\left(\frac{nt}{2}\right).$$

5.2. Method of Undetermined Coefficients. The solution to the homogeneous problem associated with (5.1) is $y_h(t) = c_1 \cos(3t) + c_2 \sin(3t)$, $c_1, c_2 \in \mathbb{R}$. Knowing this, if you were to use the method of undetermined coefficients³ then what would your choice for the particular solution, $y_p(t)$? DO NOT SOLVE FOR THE UNKNOWN CONSTANTS

Now that we know the functional form of $f(t)$ in terms of cosine functions, and a constant, we can use the method of undetermined coefficients (see diffEQ review on ticc for details) to pick the form of the particular solution. Doing so gives,

$$(5.6) \quad y_p(t) = A_0 + A_6 t \cos(3t) + B_6 t \sin(3t) + \sum_{n=1, n \neq 6}^{\infty} A_n \cos\left(\frac{nt}{2}\right) + B_n \sin\left(\frac{nt}{2}\right),$$

where the $n = 6$ term has been omitted from the sum because its fundamental frequency is the same as the homogeneous solution and therefore must be picked with a multiplier of t so that it is linearly independent of the homogeneous solution.

¹The advantage of expressing $f(t)$ as a Fourier series is its validity for any time t . An alternative approach have been to construct a solution over each interval $n\pi < t < (n+1)\pi$ and then piece together the final solution assuming that the solution and its first derivative is continuous at each $t = n\pi$.

²It is worth noting that this concepts are used by structural engineers, a sub-disciple of civil engineering, to study the effects of periodic forcing on buildings and bridges. In fact, this problem originate from a textbook on structural engineering.

³This is also known as the method of the 'lucky guess' in your differential equations text.

5.3. **Resonant Modes.** What is the particular solution associated with the third Fourier mode of the forcing function?⁴⁵

Since we have used the previous results from problem 3 we consider the sixth-mode. The sixth-mode in the Fourier series is a resonant mode, which can be seen by the particular solution it generates.

$$(5.7) \quad y_{p6}(t) = A_6 t \cos(3t) + B_6 t \sin(3t)$$

The t -prefactor causes the solution to oscillate with increasing amplitude in time. If this persists for a long time then this resonance could possibly compromise the structure that is oscillating. The point here is that any system that is prone to oscillate can be made to resonate and that this could be a serious problem. In order to understand resonance one must understand two things:

- (1) The natural frequency of the oscillator.
- (2) The frequency of the oscillatory forcing.

In our case the first of these items is straight-forward. However, it is not obvious that this periodic forcing can resonate the system until it is converted to its associated Fourier series representation.

NOTE: If you have chosen to use the 2π -periodic representation of the forcing function then you would find,

$$(5.8) \quad f(t) = \frac{\pi}{4} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi} ((-1)^n - 1) \cos(nt),$$

and now the resonant mode is at $n = 3$. Moreover, there are no even modes.

5.4. **Structural Changes.** What is the long term behavior of the solution to (5.1) subject to (5.2)? What if the ODE had the form $y'' + 4y = f(t)$?

If you were to change the spring in this mass-spring system so that its Hooke's law constant is $k = 4$ then you would also change the homogeneous solution to be $y_h(t) = c_1 \cos(2t) + c_2 \sin(2t)$ and now we can see that no term in the Fourier series representation of $f(t)$ causes a resonance. Thus, a change to the structure can avoid resonance. Of course most objects are far more complicated than a simple harmonic oscillator, which makes it difficult to understand its resonant frequencies. Regardless, the idea that one can avoid resonant behavior by changing the structure itself is a practical since it is often the case that the external forces are beyond out control.

⁴Each term in a Fourier series is called a mode. The first mode is sometimes called the fundamental mode. The rest of the modes, called *harmonics* in acoustics, are just referenced by number. The third Fourier mode would be the third term of Fourier summation

⁵Note: If you reuse the results of problem 3 from this assignment then you should consider the sixth-mode not the third. If you evaluate the Fourier series representation straight-up in terms of the 2π -periodic formula then the resonant mode turns out to be the third-mode.