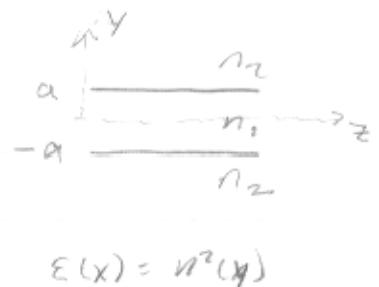


Dielectric waveguides

qualitative differences

- loss-free guiding by TIR
- evanescent wave exists in cladding
must match functions at boundaries.
- used in IR, visible, UV part of spectrum
- lowest order mode iof symmetric w.g. is not cut off.
- possible to have gradient index

EM solutions for modes. one-D slab



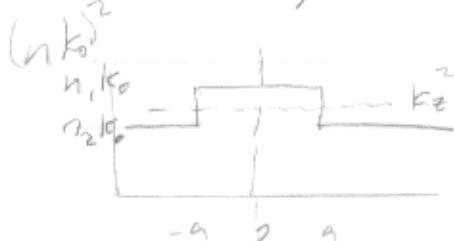
$$\frac{\partial^2 \vec{E}_T}{\partial y^2} + \epsilon(y) k_0^2 \vec{E}_T = k_z^2 \vec{E}_T$$

$$\epsilon(x) = n^2(x)$$

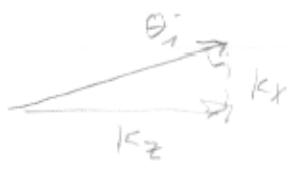
qualitative solution:

QM equivalent pot'l $U(y) \sim -n^2(y) k_0^2$

$$\text{"eigenval"} \quad E \sim -k_z^2$$



- k_z is bounded by $n_2 k_0$ to $n_1 k_0$
for guided mode.
- Lowest modes: $k_z \lesssim n_1 k_0$
cutoff $k_z \approx n_2 k_0$
- use $\sin \theta$ type solutions to estimate # modes



$$n_1 \sin \theta_c = n_2$$

$$n_1 k_z / n_1 k_0 = n_2$$

Cutoff condition

As a mode approaches cutoff, more field energy in cladding.



At cutoff, no decay in cladding: $\beta \rightarrow 0$

$$\beta^2 = k_z^2 - n_2^2 k_0^2 = 0 \rightarrow k_z^2 = n_2^2 k_0^2$$

this is same conclusion (same physics) as TIR cutoff:

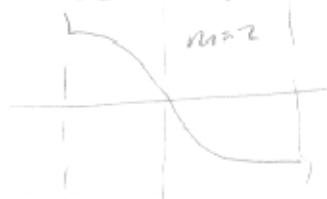
at critical angle, $k_z = k_0 n_2$, $\sin \theta_c = n_2 k_0$

$$k^2 = n_1^2 k_0^2 - k_z^2 \Rightarrow (n_1^2 - n_2^2) k_0^2 \text{ at cutoff}$$

$$= V^2/a^2$$

Since α is transverse wavenumber,

$$\alpha = 2\pi/\lambda_T$$



at cutoff

$$m \frac{\lambda_T}{2} = 2a = \frac{m}{2} \frac{2\pi}{\alpha}$$

modes

$$m_{max} = \frac{2a\alpha}{\pi} = \frac{2V}{\pi}$$

takes integer values, but m_{max} is always at least 1
single mode if $\frac{2V}{\pi} < 2$ or $\boxed{\frac{V}{\pi} < 1}$

At cutoff (dielectric case) $k_z = n_2 k_0$

i.e. in cladding $k_y = 0$, evanescent field isn't clamped

\rightarrow estimate # bound modes using metal w.g. dispersion
inside w.g. (metal)

$$k_y \cdot \frac{m\pi}{(2a)} \rightarrow k_z^2 = n_1^2 k_0^2 - \frac{m^2 \pi^2}{4a^2} \geq n_2^2 k_0^2$$

solve for m : bound if inequal. is true.

$$m^2 \leq (n_1^2 - n_2^2) k_0^2 a^2 \frac{4}{\pi^2}$$

V-number $V = (n_1^2 - n_2^2)^{1/2} k_0 a$

mode m is bound if

$$m < V \cdot 2/\pi \quad (\text{estimate})$$

estimate tends to be low by 1 or 2 modes,

\sqrt{m} # bound modes.

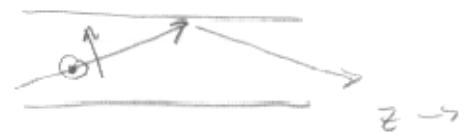
Solve for modes

$$\frac{\partial \vec{E}}{\partial y^2} = - (n_i^2 k_0^2 - k_z^2) \vec{E} \quad i=1,2$$

here, we'll solve for field that is purely transverse

TE : solve for E_x

TM : solve for B_x



most cases,

$$\frac{n_1 - m}{n_1} \ll 1 \quad \text{"weak-guiding"}$$

ignore variation in ϵ so that $\nabla \cdot (\epsilon \vec{E}) \approx \epsilon \nabla \cdot \vec{E}$
 \rightarrow modes are close to TEM

Note k_y 's depend on region, $n_{1,2}$



for $|k_y| < a$

$k_y = \text{real}$, oscill. solns

let $k_y = \alpha$ in core

$$E \sim \cos \alpha y \text{ or } \sin \alpha y$$

for $|k_y| > a$

$k_y = \text{imag}$ (evanescent wave)

let $k_y = i\beta$ in cladding. $E \sim e^{\pm \beta x}$

Take advantage of symmetry:

odd $\rightarrow \sin \alpha y$



even $\rightarrow \cos \alpha y$



match at one boundary.

TE even solns

$$E_x = E_1 \cos \alpha y \quad \text{core}$$

$$= E_2 e^{-\beta y} \quad y > a \quad \text{cladding}$$

at $y=a$ match tang. compn.

$$E_1 \cos \alpha a = E_2 e^{-\beta a}$$

second B.C. from \vec{B} $\mu = 1$ everywhere, all \vec{B} contain

$$\nabla \times \vec{E} = i \frac{\omega}{c} \vec{B}$$

$$\begin{vmatrix} n & y & z \\ x & \partial_y & \partial_z \\ \partial_x & \partial_y & \partial_z \\ E_x & 0 & 0 \end{vmatrix} = y \partial_z E_x - \frac{1}{2} \partial_y E_x = i \frac{\omega}{c} \vec{B}$$

$$\Delta B_z = 0 \rightarrow \Delta(\partial_y E_x) = 0 \therefore \text{match slope.}$$

$$E_x' \Big|_a = -\alpha E_1 \sin \alpha a \quad \text{core}$$

$$= -\beta E_2 e^{-\beta a} \quad \text{cladding}$$

divide two eqns, eliminate E_1, E_2

$$\underline{\alpha a \tan \alpha a = \beta a}$$

$$\text{odd solution} \rightarrow \underline{\alpha a \cot \alpha a = -\beta a}$$

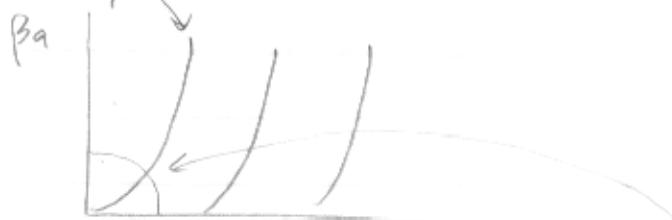
ultimately k_z is eigenvalue.

$$\alpha^2 = n_1^2 k_0^2 - k_z^2$$

$$\beta^2 = -(n_2^2 k_0^2 - k_z^2)$$

$$\text{note } \alpha^2 + \beta^2 = n_1^2 k_0^2 - n_2^2 k_0^2 \\ = V^2/a^2$$

plot βa as function of αa



also plot $\beta a = \sqrt{V^2 - (\alpha a)^2}$ circles
intersect \rightarrow modes.

numerically find roots.

Cylindrical waveguides

$$\nabla_r^2 E_r - k_z^2 E_r + n_i^2 k_o^2 E_r = 0$$

\rightarrow cyl. coords

$$\frac{1}{r} \partial_r (r \partial_r E_r) + \frac{1}{r^2} \partial_\phi^2 E_r + (n_i^2 k_o^2 - k_z^2) E_r = 0$$

$$\text{separation of variables: } E_r(r, \phi) = R(r) \Phi(\phi)$$

$$\rightarrow \frac{1}{\Phi} \partial_\phi^2 \Phi = \text{const.} \quad (\text{mult. eqn by } \frac{r^2}{R(r) \Phi(\phi)})$$

$$\Phi(\phi) = e^{im\phi} \quad m = \pm \text{integer} \\ = \text{azimuthal mode index}$$

expansion and eliminating out,

radial eqn is

$$\frac{d^2 R}{dr^2} + \frac{1}{r} \frac{dR}{dr} + \left(n_i^2 k_o^2 - k_z^2 - \frac{m^2}{r^2} \right) R = 0$$

$$\text{let } u = kr \rightarrow \frac{dR}{dr} = \frac{du}{dr} \frac{dR}{du} = kr \frac{dR}{du}$$

$$\rightarrow kr^2 \frac{d^2 R}{du^2} + \frac{k^2}{u} \frac{dR}{du} + \left(n_i^2 k_o^2 - k_z^2 - \frac{k^2 m^2}{u^2} \right) R = 0$$

$$u^2 \frac{d^2 R}{du^2} + u \frac{dR}{du} + \underbrace{\left[\left(\frac{n_i^2 k_o^2 - k_z^2}{k^2} \right) u^2 - m^2 \right]}_{= l} R = 0$$

\rightarrow Bessel equation. $J_m(u)$ regular at origin
 $N_m(u)$ diverge at origin
 both oscillatory.

modified Bessel $J_m(kr) = J_m(ikr)$ exp growing
 $K_m(kr) = N_m(ikr)$ exp damped.