## Response to FS and FT $\mathrm{Q}+\mathrm{A}$

The results are in. I have read your questions from HW8. There were many interesting questions and maybe it would be good to do again. The questions below were common and summarize important points about the material we studied in chapter 11. I will address other interesting questions in future classes.

## 1. Concerning the Parameter $L$.

There are persistent questions about how one determines $L$ in the context of Fourier coefficent calculations. First, a periodic function is a function whose graph can be created by the repetition of the graph on a principle width or domain. We take this width to be $2 L$ where $L$ is determined by the principle domain. So, if I tell you $f$ repeats itself every 15 -units then you say $L=7.5$. So, $L$ is determined from the width needed to reconstruct the graph of $f$.

Second, once $L$ is determined, the bounds of the integral must be set to 'encompass' this width. ${ }^{1}$ So, if you wanted to find the FS of a function $g$, which is has periodicity $g(x)=$ $g(x+16)$ then you would do some integrals like $a_{0}=\int_{-8}^{8} g(x) d x$. However, if you are given the definition of this function for $x \in(0,16)$ you would also find a Fourier series by doing integrals like $a_{0}=\int_{0}^{16} g(x) d x$.
Lastly, if you take the case where $L \rightarrow \infty$ then the function will 'encompass' all of $\mathbb{R}$ and thus will likely not be periodic. ${ }^{2}$ In this case the Fourier series becomes a Fourier integral and though the algorithm is very much the same the formula and purpose are quite different. If a Fourier series is used as a prescription for writing down periodic functions and functions defined on bounded domains ${ }^{3}$ then a Fourier integral is a prescription for writing down nonperiodic functions. So, Fourier methods can be used any time one wants to solve problems whose solution must be defined on bounded or un-bounded domains. ${ }^{4}$

## 2. Concerning Half-Range Expansions (Section 11.3)

There were many questions about half-range expansions and their relationship to cosine and sine series. First, we need to all agree that if a function is periodic then the function has a Fourier series representation. Second, if the periodic function has an odd symmetry then the FS has only sine-terms and is called a Fourier sine series. On the other hand, if the periodic function has an even symmetry then it will have cosine terms, possibly a $a_{0} \cos (0 \cdot x)$ term, and is called a Fourier cosine series. We now ask the question, 'Scott, I know the class of periodic functions is large, but can we make it larger?'
Well, since you asked, we can find more functions, which have Fourier series representations. RECAP! So, far we have that if $f$ is a $2 L$ periodic function then we have a procedure of representing it with sines and cosines or even with imaginary-exponentials if that's your game. Now consider the class of functions defined on a bounded domain in $\mathbb{R}$. That is, all functions $f(x), x \in(a, b), a, b \in \mathbb{R}, a<b .{ }^{5}$ Notice that $f$ has no definition outside of $(a, b)$

[^0]and that this means we are free to do whatever we wish for $f(x)$ for $x \in(-\infty, a)$ or $(b, \infty)$. Really. We can do Whatever we want so long at we make the extended function agree with $f$ on ( $a, b$ ).
There are many ways one would extend a function to the rest of the number line. ${ }^{6}$ If we repeat the function $f$ for all $b-a$ 'chunks' of the number line then you would have a periodic function and thus a FS representation. We can do better! Suppose, we construct a function that is even and then repeat this even-extension. We now have a even periodic function and thus a Fourier cosine series representation. Since, all of this is induced from the non-periodic function $f$ we call the even periodic function a cosine half-range expansion. An odd-periodic extension would give a sine half-range expansion.

So from this we conclude that not only do FS give us a prescription for writing down any function in the linear vector space of all periodic functions we have that any function defined on a bounded domain also has an equivalent FS representation on that domain. Moreover, this periodic extension is not unique and there exists both even and odd FS representations for such a function. The even extension is called a cosine half-range expansion and the odd extension is called the sine half-range expansion. This is an extremely large class of functions with a general method of representing them. ${ }^{7}$
3. Concerning Fourier Integral through Transforms 11.7-11.8

There were many exciting questions over this material. Here are some I recall:

- What does the FT give us? How should I think about the error? How is this related to the FI? Needs outside of signal processing?
- Does every function have an FT? Are there other transforms?
- How does FT help with non-periodic functions? AKA How are FT useful other than the standard frequency/amplitude shtick?
- What was the coolest thing you have done with FS? Are these done by hand anymore?
- What is convolution used for?
- What is a Fourier Mode? What would be a physical manifestation of a mode?
- What the heck was that Green's function stuff?
- What is a dirac-delta function?

These are all important questions and I hope to say more about as many of them as possible. It is too early to call how many of them I will get to. For right now we should take the following away from Fourier methods:
(a) Equation (3) gives the representation of a $2 L$-periodic function $f$ using a linear combination of the cosine/sine basis vectors. ${ }^{8}$ The Fourier coefficients are the constants in the combination and are given in terms of integrals of $f$ with the 'basis-vectors.' This integral (4) measures how much of each 'basis-vector' is needed to represent $f$ in summation.

[^1](b) If we argue that as $L \rightarrow \infty$ the function ceases to be periodic and a Fourier integral results from the Fourier series. One can see quite natural similarities between (5)-(6) and (1)-(2) but clearly the use of a continuously infinite sum instead of a discreetly infinite sum must have consequences. In this case we take (5)-(6) to be the representation of a non-periodic function using 'linear combinations' of 'more' sine/cosine vectors. ${ }^{9}$
(c) The same fists that mangled (1)-(2) into (3)-(4) can be used to mangle (5)-(6) into (9). This gives the interpretation that $\hat{f}$ are the 'Fourier coefficients' of the function $f$ and that $f$ is represented as a continuous infinite sum of sine/cosine functions. It turns out that often it is more practical to do particular calculations on the transformed function $\hat{f}$.
(d) We typically are concerned about finding the transform of a given function and less concerned about the inverse transform since it must return the original function $f$.
(e) To take the Fourier transform of a function $f(x)$ you must perform in integral on the left in equation (9). The result should only depend on the transform variable $\omega$. In the case that the function has symmetry then (7)-(8) can be seen as special cases.
(f) The signal interpretation is that if $f$ is the amplitude of the signal as a function of time then $\hat{f}$ defines the complex amplitude as a function of angular frequency. From this perspective one can calculate the 'energy' contained within the signal from the amplitudes of oscillation for particular frequency ranges. If you wanted to change these features then you would alter $\hat{f}$ and thus need to take an inverse transform toe 'hear' signal as a function of time.
(g) Physically, this gives us the ability to simultaneously consider two perspectives, $x$ and $\omega$, of a particular function and leads to a fundamental precept of quantum mechanics that localization in one variable tends to imply delocalization in the transformed space. In this way no one can know both perfectly and it is argued that the smaller you are the more this tends to apply to you. This sort of uncertainty principle tends to wash out when things are more classical.

In the future we will discuss uses of these methods but for right now we should be concerned with calculations. Particularly, given $f$ one should be able calculate $\hat{f}$.

[^2]
[^0]:    ${ }^{1}$ See homework 6 problem 3
    ${ }^{2}$ I say likely $\mathrm{b} / \mathrm{c}$ there are, of course, odd balls. If you take a constant function and do this then you will get a constant function, both of which are periodic. Booo!
    ${ }^{3}$ see half-range question
    ${ }^{4}$ It might seem that any function should fit into these categories but in the formal treatment of these tools it is assumed that the function has subtitle decay at infinity and they have finite integrals.
    ${ }^{5}$ This might occur because the object we are talking about is physical and doesn't exist outside of this domain.

[^1]:    ${ }^{6}$ I thought, http://www-solar.mcs.st-and.ac.uk/~alan/MT2003/PDE/node15.html had some nice pictures.
    ${ }^{7}$ This is similar to how Taylor series gives us a general method of representing a function in terms of its curvature/derivatives.
    ${ }^{8}$ Well, there are sines and cosines in those $e$ 's

[^2]:    ${ }^{9}$ Using the classic black magic of delta distributions (AKA dirac-delta 'functions') then one can even use this equation to recover FS. So, really what we are talking about is a generalization of (3)-(4).

