PHGN 462/507
Homework 10 due Friday 10 Nov by 5pm

1) Calculate the phase and group velocities for a wave in plasma ($\omega > \omega_p$) in the case where the collision rate is zero. For the group velocity, use $v_g = (dk/d\omega)^{-1}$. Sketch or plot these functions as a function of ω.

2) The refractive index normally rises with increasing frequency. In the neighborhood of an absorption, the dispersion is anomalous.
   a. Find the width of this anomalous region (where $dn/d\omega < 0$) for the region around an absorption at ω_0. Assume that $\beta << \omega_0$.
   b. Show that the index of refraction assume its minimum and maximum values at points where the absorption coefficient is at half-maximum.

3) Calculate the Fourier transform of the function $\cos^2(\omega_0 t)$. Sketch the result. If you have a delta function $\delta(\omega)$, show it as a spike with unit height, $a\,\delta(\omega)$ would be a spike with height a.

4) Use the convolution theorem and any other theorems to calculate the Fourier transform of the function $E(t) = E_0 \sin^2 \omega_0 t \exp(-a t^2)$. Show all your work: you shouldn't have to do any integrals; make use of the theorems and transform pairs. Sketch or plot the spectrum ($|E(\omega)|^2$)in the limit that $\omega_0 >> a$.

5) A Gaussian pulse of light travels a distance z through a medium with a dispersive index of refraction $n(\omega)$. The input field is $E_{in}(t) = E_0 e^{-t^2/\tau_0^2} e^{-i\omega_0 t}$
   The spectral phase imparted by the medium is $\phi(\omega) = \omega\, n(\omega) z / c$.
   a. Taylor expand the spectral phase around ω_0, and keep only the second order term. (This means that our frame of reference moves with the pulse at the group velocity). Show that the output spectrum can be written as
   $$\tilde{E}_{out}(\omega) = E_0 \sqrt{\pi \tau_0^2} \exp\left[-\left((\omega - \omega_0)^2 \tau_0^2 / 4\right)\left(1 - i\Gamma(z)\right)\right],$$
   where $\Gamma(z) = 2\phi''(\omega_0, z)/\tau_0^2$ and $\phi''(\omega_0, z) = \partial^2 \phi / \partial \omega^2 \big|_{\omega_0}$.
   b. Using the transform theorems (shift, scale), calculate the output electric field in the time domain, and show that it can be put into the form:
   $$E_{out}(t) = \frac{E_0}{\sqrt{\tau(z)/\tau_0}} \exp\left[-t^2/\tau^2(z)\right] \exp\left[-i\left(\omega_0 t + b(z) t^2\right)\right],$$
   where the pulse duration is now a function of z: $\tau(z) = \tau_0 \sqrt{1 + (z/z_c)^2}$,
   and the pulse has a linear chirp (instantaneous frequency $\omega_{inst} = -\partial\phi(\omega)/\partial\omega$) varies in time) with a chirp parameter: $b(z) = \dfrac{1}{\tau_0^2} \dfrac{z z_c}{z^2 + z_c^2}$.

Here, $z_c \equiv \left(\frac{2}{c\tau_0^2} \frac{\partial^2}{\partial\omega^2} (\omega\, n(\omega)) \Big|_{\omega_0} \right)^{-1}$ is the characteristic distance over which the pulse broadens.

c. For glass at a wavelength of 800nm, $n(\omega_0) = 1.51$, $n'(\omega_0) = 6.7 \times 10^{-3}\, fs$, and $n''(\omega_0) = -4.9 \times 10^{-5}\, fs^2$. Calculate z_c for a pulse duration of τ_0=10fs, and plot $\tau(z)$ and $b(z)$ for a distance of z/z_c from 0 to 5.