Quasi-monochromatic light, pulses, and Fourier analysis

So far whenever we've been dealing with radiation we've thought about it one frequency at a time - we've been looking at monochromatic light, such as your idealized monochromatic plane wave, which looks like

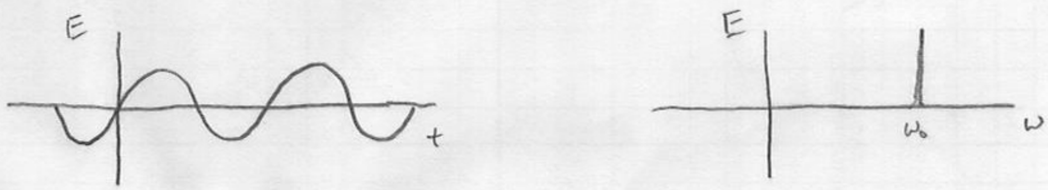
$$\vec{E}(\vec{x},t) = \vec{E}_0 e^{i(\vec{k}\cdot\vec{x} - \omega_0 t)}$$

where $\vec{E}_0$ and ω_0 are constants

For much of what follows, it's useful to hold position fixed and just look at the field's time dependence, so we strip out $\vec{x}$ (and the polarization info too, while we're at it):

$$E(t) = E_0 e^{-i\omega_0 t}$$

Now let's expand things just a bit and consider sources that are quasi-monochromatic - which is to say, almost but not quite monochromatic. We know from Fourier theory that only a pure, infinite sine wave can be <u>strictly</u> monochromatic. Remember, we can find the frequency spectrum of a source by taking the Fourier transform of the time-domain field, and vice versa, and the Fourier transform of a sine wave is a delta function, and vice versa:



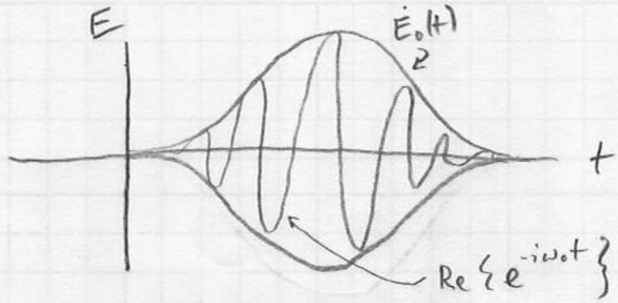
time-domain representation of a monochromatic source of frequency ω_0

frequency-domain representation of that same source

A quasi-monochromatic source could include one that still has the same <u>carrier frequency</u> ω_0, but instead of being a sine wave that goes on forever, is shaped by some kind of overall envelope.

$$E(t) = \underset{\substack{\uparrow \\ \text{envelope function, sometimes called modulation}}}{E_0(t)} \, \underbrace{e^{-i\omega_0 t}}_{\text{carrier wave}}$$

This might look like (in the time domain):



What does this thing's frequency-domain representation look like? Let's recall how to do a Fourier transform. There are a few different definitions, differing in where they stick the 2π factors. We'll be using the following convention & notation:

Fourier transform: $\mathcal{F}(E(t)) = \mathcal{E}(\omega) = \int_{-\infty}^{\infty} E(t)\, e^{i\omega t}\, dt$

time-domain E-field

Fourier transform operation

frequency-domain E-field

watch signage

Inverse transform: $\mathcal{F}^{-1}(\mathcal{E}(\omega)) = E(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{E}(\omega)\, e^{-i\omega t}\, d\omega$

2π goes in inverse transform only

Our quasi-monochromatic pulse shown above probably has a frequency spectrum centered on ω_0. To find out what else is in there, first we write a complete time-domain field representation. Pulses with Gaussian profiles are pretty common in laser physics, so let's let the envelope be

$$E_0(t) = E_0\, e^{-t^2/\tau^2} \quad \text{for some constant } \tau$$

Then

$$E(t) = E_0\, e^{-t^2/\tau^2}\, e^{-i\omega_0 t}$$

and we want to take the Fourier transform of that.

Side note: As soon as you start doing Fourier transforms of fields, you start being really glad that you have a system for representing those fields as complex exponentials instead of trig functions.

$$\mathcal{E}(\omega) = \int_{-\infty}^{\infty} E_0 e^{-t^2/\tau^2} e^{-i\omega_0 t} e^{i\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} E_0\, e^{-t^2/\tau^2} e^{i(\omega-\omega_0)t}\, dt$$

You can jam this into Mathematica, but there's a cute way of doing it by hand without much trouble.

You may recall that this definite integral $\int_{-\infty}^{\infty} e^{-x^2} dx$

is a pretty fundamental one that evaluates to $\sqrt{\pi}$
(also recall that e^{-x^2} has no antiderivative, so you can only do it as a definite integral using some cleverness)

We'll put our integral into that form by completing a square

$$e^{-t^2/\tau^2} e^{i(\omega-\omega_0)t} = e^{-t^2/\tau^2 + i(\omega-\omega_0)t}$$

So let's look for a good b such that $e^{-(t/\tau + b)^2}$

Note that $e^{-(t/\tau + b)^2} = e^{-[t^2/\tau^2 + b^2 + 2bt/\tau]}$

So we might try letting $\dfrac{2bt}{\tau} = -i(\omega-\omega_0)t \Rightarrow b = \dfrac{-i(\omega-\omega_0)\tau}{2}$

in which case $e^{-(t/\tau + b)^2} = e^{-(t/\tau - i(\omega-\omega_0)\tau/2)^2} = e^{-t^2/\tau^2} e^{i(\omega-\omega_0)t} e^{+\frac{(\omega-\omega_0)^2\tau}{4}}$

Which is almost our original integrand; we just need to cancel that last term on the end:

$$\mathcal{E}(\omega) = \int_{-\infty}^{\infty} E_0\, e^{-t/\tau^2} e^{i(\omega-\omega_0)t}\, dt = \int_{-\infty}^{\infty} E_0\, e^{-(t/\tau - i(\omega-\omega_0)\tau/2)^2} e^{-(\omega-\omega_0)^2\tau^2/4}\, dt$$

$$= E_0\, e^{-(\omega-\omega_0)^2\tau^2/4} \int_{-\infty}^{\infty} e^{-u^2}\, dt$$

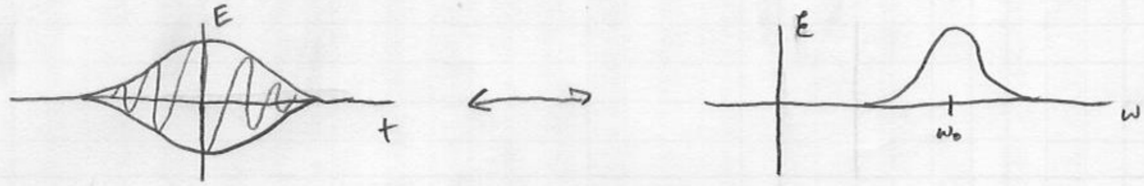
With $u = t/\tau - i(\omega-\omega_0)\tau/2$, so $du = 1/\tau\, dt$ and

$$\mathcal{E}(\omega) = \frac{E_0}{\tau} e^{-(\omega-\omega_0)^2\tau^2/4} \overbrace{\int_{-\infty}^{\infty} e^{-u^2}\, du}^{\sqrt{\pi}}$$

$$\Rightarrow \quad \mathcal{E}(\omega) = \frac{E_0\sqrt{\pi}}{\tau} e^{-(\omega-\omega_0)^2\tau^2/4}$$

$$E(t) = E_0 e^{-t^2/\tau^2} e^{-i\omega_0 t}$$

So the frequency spectrum associated with a gaussian pulse is also gaussian, centered on the carrier frequency ω_0



Remember, even though I use E for the time domain and $\mathcal{E}$ for the frequency domain, these are both perfectly valid representations of the same E-field.

Also, look at our expressions for $E(t)$ + $\mathcal{E}(\omega)$. There's an interesting piece of physics in there. τ is basically the temporal width of the pulse, and while τ shows up in the denominator for E, it's in the numerator for $\mathcal{E}$. There's a reciprocal relationship there, and it turns out to work for non-Gaussian pulses, too: A signal that's wider temporally is narrower spectrally, and vice versa.

Put another way, if you wanted to make, for example, a very short pulse of light, that pulse would have to contain many, many colors. And to get a truly monochromatic light source, only an infinitely long (temporally) signal would do.