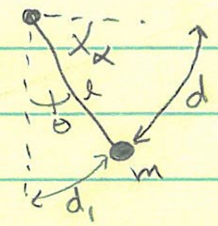
PH 350 Fall 2011 Practice Test 2 solns

1) a) There is only 1 degree of freedom, described by the angle as the pendulum swings back and forth.

b) θ, α, $d_1 = l\theta$ or $d_2 = l\alpha$ would all give you relatively simple lagrangians of only 1 coordinate. Not and! Only 1 coordinate is used.



c) Use θ: $x = l\sin\theta$; $y = -l\cos\theta + v_{oy}t - \frac{1}{2}at^2$

$\dot{x} = l\cos\theta\,\dot{\theta}$; $\dot{y} = +l\sin\theta\,\dot{\theta} + v_{oy} - at$

$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 = \frac{1}{2}ml^2\dot{\theta}^2 + 2(v_{oy}-at)l\sin\theta\,\dot{\theta} + (v_{oy}-at)^2$

$U = mgy = -mgl\cos\theta + mgv_{oy}t - \frac{1}{2}mgat^2$ {I used $\sin^2\theta + \cos^2\theta = 1$}

$$\rightarrow \boxed{L = \frac{1}{2}ml^2\dot{\theta}^2 + 2(v_{oy}-at)l\sin\theta\,\dot{\theta} + (v_{oy}-at)^2 + mgl\cos\theta - mgv_{oy}t + \frac{1}{2}mgat^2}$$

d) $H = \frac{\partial L}{\partial\dot{\theta}}\dot{\theta} - L = \left(ml^2\dot{\theta} + 2(v_{oy}-at)l\sin\theta\right)\dot{\theta} - L$

$$\rightarrow \boxed{H = \frac{1}{2}ml^2\dot{\theta}^2 - mgl\cos\theta - (v_{oy}-at)^2 + mgv_{oy}t - \frac{1}{2}mgat^2}$$

To get $H(\theta, p_\theta)$ {I could ask that on the test}, $p_\theta = \frac{\partial L}{\partial\dot{\theta}} = ml^2\dot{\theta} + 2(v_{oy}-at)l\sin\theta$

$\rightarrow \dot{\theta} = \frac{p_\theta}{ml^2} - 2(v_{oy}-at)l\sin\theta$

$$\rightarrow \boxed{H(\theta, p_\theta) = \frac{1}{2}ml^2\left[\frac{p_\theta}{ml^2} - 2(v_{oy}-at)l\sin\theta\right]^2 - mgl\cos\theta - (v_{oy}-at)^2 + mgv_{oy}t - \frac{1}{2}mgat^2}$$

e) $\frac{dH}{dt} = \frac{\partial H}{\partial t} \neq \emptyset$ $\rightarrow$ No it's not conserved.

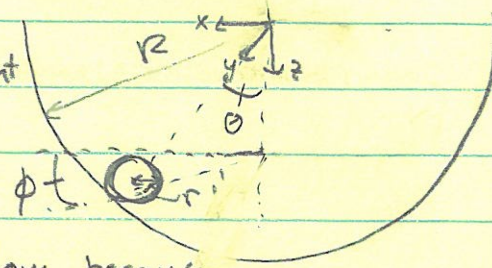
f) No, because $y(\theta; t)$ depends on t.

g) I don't like (g) as much as I used to.

<u>Practice Test 2 solns pg. 2</u>

2) a) 6 : 3 rotational and 3 translational.

b) An object constrained to any 2-D surface is constrained in one translational direction (it can't come off the surface)
→ <u>1 translational constraint.</u>



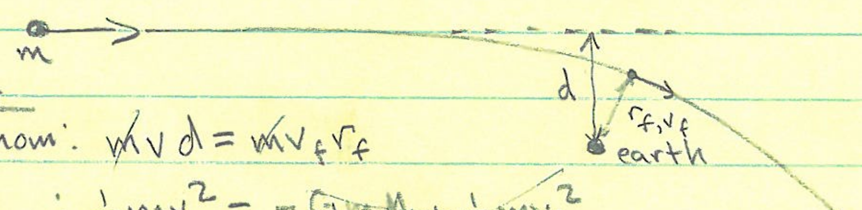
This could be written as the single constraint equation $x_{cm}^2+y_{cm}^2+z_{cm}^2=(R-r)^2$

Rotationally you lose all degrees of freedom because if it's not slipping then $\omega_{\perp \text{ plane of motion}} = \frac{v}{r}$ and the two components of ω in the plane of motion $= \phi$. → 3 constraints.
<u>3 rotational constraints</u>

$\boxed{4 \text{ constraints}}$ → $\boxed{2 \text{ deg. of freedom}}$

c) See the above picture: I would pick the standard θ, ϕ from spherical coordinates. Anything proportional to those will be identical. If you chose something different, be sure you motivate why you chose it. For instance my coords. give simple transformations: $x=(R-r)\sin\theta\cos\phi$; $y=(R-r)\sin\theta\sin\phi$
$z=(R-r)\cos\theta$

3)



Cons. of ang. mom: $\cancel{m} v d = \cancel{m} v_f r_f$

Note! I find out this is wrong later.

Energy: $\underset{\uparrow \; U=0 @ \infty}{\frac{1}{2}mv^2} = -\frac{GmM}{r_f^2} \cancel{+ \frac{1}{2}mv_f^2}$

$\rightarrow r_f = \frac{vd}{v_f} \rightarrow \frac{1}{2}mv^2 = -\frac{GmM}{v^2d^2/v_f^2} + \frac{1}{2}mv_f^2 = \cancel{\left(\frac{-GmM}{v^2d^2}+\frac{1}{2}m\right)v_f^2}$

(see next page)

Practice Test 2 solns pg. 3

3 cont'd) $\rightarrow \frac{1}{2}\cancel{m}v^2 = \left(-\frac{G\cancel{m}M}{v^2d^2} + \frac{1}{2}\cancel{m}\right)v_f^2$

$$\cancel{\boxed{\rightarrow v_f^2 = \frac{v^2}{\left(1 - \frac{2GM}{v^2d^2}\right)}}}$$

or $\cancel{\boxed{\rightarrow v_f = \frac{v}{\sqrt{1 - \frac{2GM}{v^2d^2}}}}}$ $\rightarrow r_f = \frac{vd}{v_f} = d\sqrt{1 - \frac{2GM}{v^2d^2}}$

Units: $\frac{GM}{d^2} = [m/s^2] \rightarrow \frac{GM}{v^2d^2} = \left[\frac{1}{m}\right]$ doesn't work. Hmm.

Oops $U \neq -\frac{GmM}{r^2} \rightarrow U = -\frac{GmM}{r}$. Starting over

$\rightarrow \frac{1}{2}mv^2 = -\frac{GmM}{r_f} + \frac{1}{2}mv_f^2$; $r_f = \frac{vd}{v_f}$ {this was still good}

$\rightarrow \frac{1}{2}mv^2 = -\frac{GmM}{vd/v_f} + \frac{1}{2}mv_f^2 = -\frac{GmM}{vd}v_f + \frac{1}{2}mv_f^2$

$\rightarrow v_f^2 - \frac{2GM}{vd}v_f - v^2 = \emptyset$

$$\rightarrow v_f = \frac{\frac{2GM}{vd} \pm \sqrt{\frac{4G^2M^2}{v^2d^2} + 4v^2}}{2} = \frac{GM}{vd} \pm \sqrt{\frac{G^2M^2}{v^2d^2} + v^2}$$

$$\boxed{v_f = \frac{GM}{vd}\left(1 \pm \sqrt{1 + \frac{v^4d^2}{G^2M^2}}\right) ; \; r_f = \frac{vd}{v_f}}$$

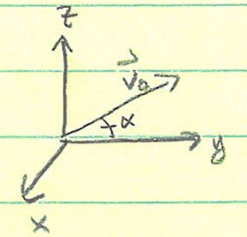
Now units:

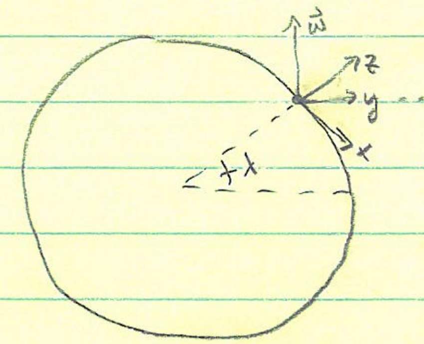
$\frac{GM}{d^2} = [m/s^2] \rightarrow \frac{G^2M^2}{d^2} = \left[\frac{m^2}{s^4} \cdot m^2\right] \rightarrow \frac{v^2d^2}{G^2M^2} = [1]$ ✓ $\frac{GM}{vd} = \left[\frac{m}{s^2} \cdot m \cdot \frac{s}{m}\right] = \left[\frac{m}{s}\right]$ ✓

+ or −? The $\sqrt{\ }$ is alway > 1 so for $v_f > \emptyset$, it must be +. $\boxed{\text{Use + root}}$

4)

$$\vec{a}_r = \vec{g} - 2\vec{\omega}\times\vec{v}_r$$





$$\vec{v}_r \sim v_0\cos\alpha\,\hat{e}_y + (v_0\sin\alpha - gt)\hat{e}_z$$

$$\vec{\omega} = -\omega\cos\lambda\,\hat{e}_x + \omega\sin\lambda\,\hat{e}_z$$

$$\vec{\omega}\times\vec{v}_r = -\omega v_0\cos\lambda\cos\alpha\,\hat{e}_z + \omega\cos\lambda(v_0\sin\alpha - gt)\hat{e}_y - \omega v_0\cos\alpha\sin\lambda\,\hat{e}_x$$

$$\to \vec{a}_r \sim (-g - \omega v_0\cos\lambda\cos\alpha)\hat{e}_z + (2\omega v_0\cos\alpha\sin\lambda)\hat{e}_x - 2\omega\cos\lambda(v_0\sin\alpha - gt))\hat{e}_y$$

$$a_{r,z} \sim -g \quad\to\quad v_{r,z} = v_0\sin\alpha - gt; \quad z(t) = v_0\sin\alpha\, t - \tfrac{1}{2}gt^2$$

$$t_f = t(z=\emptyset) = \frac{2v_0\sin\alpha}{g}$$

$$d = x(t_f) = \omega v_0\cos\alpha\sin\lambda\, t_f^2$$

$$\to d = \omega v_0\cos\alpha\sin\lambda\left(\frac{4v_0^2\sin^2\alpha}{g^2}\right)$$

$$\to \boxed{d = \frac{4v_0^3\omega}{g^2}\cos\alpha\sin^2\alpha\sin\lambda}$$