

Fourier Transforms: Transform pairs, theorems in (x, y) and (β_x, β_y) domains

Definitions and theorems (in Mathematica, use FourierParameters->{1,-1}):

Spatial frequencies: $\beta_x = kX / R = 2\pi X / \lambda R = k \sin \theta_x$.

At focal plane of lens, $R \rightarrow f$, where f = focal length of lens

$$\textbf{Forward transform (1D): } \mathfrak{F}\{g(x)\} \equiv G(\beta_x) = \int_{-\infty}^{\infty} g(x) \exp[-i\beta_x x] dx$$

$$\textbf{Inverse transform (1D): } \mathfrak{F}^{-1}\{G(\beta_x)\} = g(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\beta_x) \exp[i\beta_x x] d\beta_x$$

$$\textbf{Forward transform (2D): } \mathfrak{F}\{g(x, y)\} \equiv G(\beta_x, \beta_y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) \exp[-i(\beta_x x + \beta_y y)] dx dy$$

$$\textbf{Inverse transform(2D): } \mathfrak{F}^{-1}\{G(\beta_x, \beta_y)\} = g(x, y) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(\beta_x, \beta_y) \exp[i(\beta_x x + \beta_y y)] d\beta_x d\beta_y$$

$$\textbf{Shift Theorem: } \mathfrak{F}\{g(x - x_0)\} = \exp(-i\beta_x x_0) G(\beta_x) \quad \mathfrak{F}^{-1}\{G(\beta_x - \beta_{x0})\} = \exp(i\beta_{x0} x) g(x)$$

$$\textbf{Scale Theorem: } \mathfrak{F}\{g(ax)\} = \frac{1}{|a|} G(\beta_x / a) \quad \mathfrak{F}^{-1}\{G(b\beta_x)\} = \frac{1}{|b|} f(x/b)$$

$$\textbf{Conjugate: } \mathfrak{F}\{g^*(x)\} = G^*(-\beta_x)$$

$$\textbf{Inverse transform pair: } \mathfrak{F}\{G(x)\} = g(-\beta_x) \quad \mathfrak{F}^{-1}\{g(\beta_x)\} = G(-x)$$

$$\textbf{Convolution: } h(x) = f(x) \otimes g(x) = \int_{-\infty}^{\infty} f(x') g(x - x') dx'$$

$$\textbf{Convolution w/delta fcn: } \delta(x - x_0) \otimes g(x) = g(x - x_0)$$

$$\textbf{Convolution theorem: } f(x) \otimes g(x) = \mathfrak{F}^{-1}\{F(\beta_x) G(\beta_x)\} \quad \mathfrak{F}\{f(x) g(x)\} = \frac{1}{2\pi} F(\beta_x) \otimes G(\beta_x)$$

$$\textbf{Parseval's theorem (conservation of energy): } \int |g(x)|^2 dx = \frac{1}{2\pi} \int |G(\beta_x)|^2 d\beta_x$$

Transform pairs:

Delta functions:

$$\mathfrak{F}\{\exp[\pm i\beta_{x0} x]\} = 2\pi \delta(\beta_x \pm \beta_{x0}) \quad \mathfrak{F}^{-1}\{\exp[\pm i\beta_x x_0]\} = \delta(x \mp x_0)$$

$$\textbf{Gaussian: } \mathfrak{F}\{\exp(-x^2/x_w^2)\} = \sqrt{\pi x_w^2} \exp(-x_w^2 \beta_x^2 / 4)$$

$$\textbf{Rect function (rect(u) = 1 for } -\frac{1}{2} < u < \frac{1}{2}): \quad \mathfrak{F}\{\text{rect}(x/x_0)\} = x_0 \text{sinc}(\beta_x x_0 / 2) \\ \mathfrak{F}\{\text{sinc}(x/2x_0)\} = 2\pi x_0 \text{rect}(\beta_x x_0)$$

$$\textbf{Cosine function: } \mathfrak{F}\{\cos(\beta_{x0} x)\} = \pi [\delta(\beta_x - \beta_{x0}) + \delta(\beta_x + \beta_{x0})]$$

$$\textbf{Array comb}(x/x_0) \equiv \sum_{n=-\infty}^{\infty} \delta(x - nx_0): \quad \mathfrak{F}\{\text{comb}(x/x_0)\} = (2\pi/x_0) \text{comb}[\beta_x / (2\pi/x_0)]$$

$$\textbf{Circ function: } (\text{circ}(r/a) = 1 \text{ for } r < a) \quad \mathfrak{F}\{\text{circ}(r/a)\} = \frac{2\pi a J_1(a\rho)}{\rho} = \pi a^2 jinc(a\rho),$$

where $\rho = \sqrt{\beta_x^2 + \beta_y^2}$, and $jinc(x) = 2J_1(x)/x$