Physics 311, Fall 2005

Problem Set #9

Fourier sin, cos*, and complex exponential series*

Due: Wednesday, November 9

Quote of the Problem Set:
"It is not uncommon for engineers to accept the reality of phenomena that are not yet understood, as it is very common for physicists to disbelieve the reality of phenomena that seem to contradict contemporary beliefs of physics"— H. Bauer

The more applicable the math, the more diverse (read: non-book) the problems. With the Fourier *transform* under our belts [soon!], many interesting numerical problems will become available. Meanwhile, these are meant to extend what's in the book.

Read Chapter 7 §§5-12, then do:

1. §5, Problem 11
2. §8, Problem 12
3. §9, Problem 2
4. *Hearing the shape of a function*
   As $n \to \infty$ the Fourier coefficients a_n and b_n always approach zero at least as fast as c/n where c is a constant independent of n. To be precise, here's a little theorem:

   > If a function $f(x)$ and its first k derivatives satisfy the Dirichlet conditions and are everywhere continuous, a_n and b_n approach zero at least as fast as $1/n^{k+2}$ for large n. If the $(k+1)$st derivative of f is *not* everywhere continuous, a_n or b_n (or both) approach zero no faster than $1/n^{k+2}$ for large n.

   (a) A periodic function $f(x)$ has Fourier coefficients

   $$a_n = \frac{n\pi}{n^4 + \pi^4} \quad (1)$$
   $$b_n = \frac{\pi}{n^2 + n\pi + \pi^2}. \quad (2)$$

   What can you say about the continuity of $f(x)$ and its derivatives?

   (b) If $f(x) = x - x^3$ for $-1 \le x \le 1$, *without* computing them, how fast will the Fourier coefficients decrease with n for large n?

   (c) Compute the a_n and b_n using *Mathematica* and determine if your results are consistent with part (b).
5. *World's slowest converging series for* π
   (a) Find the Fourier series for the square wave function

   $$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ 1, & 0 < x < +\pi \end{cases}. \quad (3)$$

   (b) From your result in (a), show that

   $$\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\right). \quad (4)$$

   See `mathworld.wolfram.com/PiFormulas.html` for rapidly-converging expressions; some add 14 additional figures to π with each term in the series!

6. *Summing a Fourier series*
If it were possible to measure the Fourier coefficients of a function, we might plausibly wish to sum the Fourier series to identify the original function. This is seldom easy, alas. Sometimes, though, by recognizing a series we *can* do this.

Show that

$$\sum_{n=1}^{\infty} \frac{\sin nx}{n} = \begin{cases} +\frac{1}{2}(\pi - x), & 0 < x \le \pi \\ -\frac{1}{2}(\pi + x), & -\pi \le x < 0 \end{cases} \tag{5}$$

Hints: (i) Write the sin in terms of complex exponentials; (ii) Remember the series for $-\ln(1-u)$; (iii) Remember the useful trick $1 - e^{ix} = e^{i\frac{x}{2}}\left(e^{-i\frac{x}{2}} - e^{i\frac{x}{2}}\right)$.

7. *Fourier series as best fit for specified number of terms*
When doing a 'least squares fit' of experimental data (a set of n_{pts} data pairs: values y^{obs} at a set of x values x_i) to a fitting form $y^{\text{fit}}(x)$, we invariably use the error measure

$$W \equiv \sum_{i=1}^{n_{pts}} \left[y^{\text{fit}}(x_i) - y_i^{\text{obs}}\right]^2 . \tag{6}$$

As you know, one identifies the 'best fit' parameters by setting $\partial W/\partial(\text{parameter}) = 0$, thereby minimizing W.

If an odd function $f(x)$ with period 2π is to be approximated by a Fourier sin series truncated to only m terms, we'd define the error similarly, as

$$W_m \equiv \int_{-\pi}^{+\pi} dx \, \left(f(x) - \sum_{n=1}^{m} b_n \sin nx\right)^2 . \tag{7}$$

By setting the derivatives of W_m with respect to the coefficients b_n to zero, find the values of the b_n which minimize this least squares function, and compare with the 'official' definition of the b_n. Think about the corresponding polynomial fit you did in a previous problem set and comment.