Lecture 17: Classical vs. special relativity

In physics, sometimes point of view is allowed to matter. Observers in different reference frames can disagree on the velocity of an object and not really hurt anything.

Sometimes point of view cannot matter. You cannot, for example, have two observers disagree on the acceleration of an object. Either the car accelerates, crashes into a tree, and bursts into flame, or it does not. They cannot be simultaneously true.

This is encoded in one of the most fundamental postulates of all of physics, the postulate of relativity:

"The laws of physics are the same for observers in every inertial reference frame"

This is a postulate of special relativity, but it is also a postulate of classical (or Galilean) relativity. It is four hundred years old, and Einstein did not invent it.

1632, Galileo Galilei:

"Shut yourself up with some friend in the largest room belowdecks of some large ship and there procure gnats, flies, and such other small winged creatures. Also get a great tub full of water and put within it certain fishes; let also a certain bottle be hung up, which drop by drop lets forth its water into another narrow-necked bottle placed underneath. Then, the ship lying still, observe how those small winged animals fly with like velocity towards all parts of the room; how the fishes swim indifferently towards all sides; and how the distilling drops all fall into the bottle placed underneath. Having observed all these particulars, though no man doubts that, so long as the vessel stands still, they ought to take place in this manner, make the ship move with what velocity you please, so long as the velocity is uniform and not fluctuating this way and that. You shall not be able to discern the least alteration in all the aforementioned effects, nor can you gather by any of them whether the ship moves or stands still."

Mathematically, what it actually looks like for laws to be the same in every inertial reference frame is as follows:

Here's a law, as seen by some guy in some reference frame:

$$\bar{F} = m\bar{a}$$

We should find that in another frame, an observer will measure forces, masses, and accelerations and find that

$\vec{F}' = m'\vec{a}'$ (prime indicates quantities as measured in the other frame)

There are rules for getting from the coordinates describing one frame to those describing another moving with speed $+v$ along the original's x axis:

$$\begin{aligned} x' &= x - vt \\ y' &= y \\ z' &= z \\ t' &= t \end{aligned}$$

These are the Galilean coordinate transforms

Let us transform $\bar{F} = m\bar{a}$ directly, using the above to carry us from one frame to another. Componentwise,

$$\begin{aligned} F_x &= m a_x \\ F_y &= m a_y \\ F_z &= m a_z \end{aligned}$$

$a_x = \frac{dv_x}{dt}$ We want to substitute in for the pieces of this equation in terms of primed quantities. Well, since $t = t'$, d/dt and d/dt' are the same operator. **

Thus $a_x = \frac{dv_x}{dt'}$

What is v_x in terms of v_x'? Remember, the goal is to rewrite $F=ma$ in primed coordinates and see if it looks the same.

$v_x = \frac{dx}{dt} = \frac{d}{dt}(x' + vt)$ and $x = x' + vt$

$= \frac{dx'}{dt} + \frac{d}{dt}(vt)$ The second term is v as long as v is not a function of t (definition of inertial reference frame).

$= \frac{dx'}{dt'} + v \Rightarrow$

$$\begin{aligned} v_x &= v_x' + v \\ v_y &= v_y' \\ v_z &= v_z' \end{aligned}$$

Similarly,

And if $a_x = \frac{dv_x}{dt'} = \frac{d}{dt'}\left(v_x' + v\right) = \frac{dv_x'}{dt'}$, then $a_x = a_x'$ and $a_y = a_y'$, $a_z = a_z'$

So $\bar{F} = m\bar{a}'$.

If we also assume mass is invariant under Galilean transforms,

$\bar{F} = m'\bar{a}' = \bar{F}'$

** This step isn't quite as automatic as it looks; see appendix notes

What'd we just do? We showed that in a universe governed by Galilean coordinate transforms, every observer measures the same acceleration for an object, the same net force on it, and that force, mass, and acceleration are related the same way in every frame.

This holds for other laws, too. Under Galilean transforms, everyone agrees on the force between two massive bodies. It's <u>always</u> given by GmM/r^2

Now, mostly that's because things like r, G, m, and M are identical in every frame. But you might imagine a law like, say,

$\nabla \times \bar{E} = -\partial \bar{B}/\partial t$, where perhaps in another frame $\nabla' \times \bar{E}' = -\partial \bar{B}'/\partial t'$ even though

$\bar{E}' \neq \bar{E}$ and $\bar{B}' \neq \bar{B}$

We call these laws covariant. They retain their form under coordinate transforms. <u>Laws need to do this</u>. You can't, for example, have one guy say $\nabla \cdot \bar{E} = \rho/\epsilon_0$ and another guy measure that $\nabla' \cdot \bar{E}' = \rho/\epsilon_0 - \bar{v} \cdot (\bar{v} \times \bar{B}')$

Let's check whether the Maxwell equations are actually covariant.

Consider, for example, $\nabla \cdot \vec{B} = 0$ We'll transform this into the primed frame, by writing ∇ in terms of ∇', etc.

$\nabla \cdot$ involves a bunch of single spatial derivatives.
If $y' = y$, $z' = z$, certainly $\frac{\partial}{\partial y'} = \frac{\partial}{\partial y}$ and $\frac{\partial}{\partial z'} = \frac{\partial}{\partial z}$

And if $x = x' + vt$, $\frac{\partial f}{\partial x'} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial x'}$ And $\frac{\partial x}{\partial x'} = \frac{\partial}{\partial x'}(x' + vt) = 1$

So $\frac{\partial f}{\partial x'} = \frac{\partial f}{\partial x}$; $\frac{\partial}{\partial x'} = \frac{\partial}{\partial x}$

So $\nabla \cdot$ must be $\nabla' \cdot$, and $\nabla \times$ is $\nabla' \times$, etc
Space derivative operators are unaffected by Galilean transforms.

So we know $\nabla \cdot \vec{B} = 0$ implies $\nabla' \cdot \vec{B} = 0$

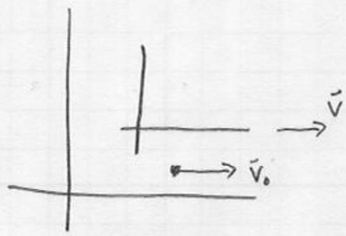
What are $\vec{E}, \vec{B}$ in terms of $\vec{E}', \vec{B}'$, classically?
That's the tricky part. Galilean transforms don't say a whole bunch about $\vec{E}$ & $\vec{B}$.

But they do say forces are invariant so

$$\vec{F} = q(\vec{E} + \vec{v}_0 \times \vec{B}) = \vec{F}' = q'(\vec{E}' + \vec{v}_0' \times \vec{B}')$$

where $\vec{v}_0'$, $\vec{v}_0$ are velocities of particles in the primed and unprimed frame, <u>not</u> the frame velocity $\vec{v}$

Let's go ahead and also assume that $q = q'$



If an object is moving at velocity $\vec{v}_0$ in the unprimed frame, and the primed frame is moving with velocity $\vec{v}$, $\vec{v}_0' = \vec{v}_0 - \vec{v}$

Thus $\quad \vec{E} + \vec{v}_0 \times \vec{B} = \vec{E}' + \vec{v}_0' \times \vec{B}' = \vec{E}' + (\vec{v}_0 - \vec{v}) \times \vec{B}'$

$$\Rightarrow \quad \vec{E} + \vec{v}_0 \times (\vec{B} - \vec{B}') = \vec{E}' - \vec{v} \times \vec{B}'$$

Now, $\vec{v}_0$ can be anything, whereas all other terms are fixed, so for this equality to hold, $\vec{v}_0 \times (\vec{B} - \vec{B}')$ must be zero for all $\vec{v}_0$, so

$\vec{B} = \vec{B}'$

Which then immediately yields $\vec{E}' = \vec{E} + \vec{v} \times \vec{B}$

So if $\vec{\nabla} \cdot \vec{B} = 0$, and $\vec{\nabla} \cdot$ is $\vec{\nabla}' \cdot$, and $\vec{B} = \vec{B}'$, then

$\vec{\nabla} \cdot \vec{B} = 0 \Rightarrow \vec{\nabla}' \cdot \vec{B}' = 0$, and so that Maxwell eqn is covariant under Galilean transforms

How about $\nabla \cdot \vec{E} = \rho/\epsilon_0$? Well, $\rho = \frac{\Delta q}{\Delta V}$ for small Δ's, and $q = q'$, and $\Delta x = \Delta x'$, $\Delta y = \Delta y'$, etc,

So $\rho = \rho'$. And $\nabla \cdot$ is $\nabla' \cdot$. And $\vec{E} = \vec{E}' - \vec{v} \times \vec{B}'$

So $\quad \nabla \cdot \vec{E} = \rho/\epsilon_0 \Rightarrow \nabla' \cdot (\vec{E}' - \vec{v} \times \vec{B}') = \rho'/\epsilon_0$

$$\Rightarrow \quad \nabla' \cdot \vec{E} = \rho'/\epsilon_0 + \nabla' \cdot (\vec{v} \times \vec{B}')$$

Which only reproduces Gauss's law in the primed frame if

$\nabla' \cdot (\vec{v} \times \vec{B}') = 0$ for all $\vec{v}$, $\vec{B}'$. (Hint: this isn't true for all $\vec{v}, \vec{B}'$)

So that's kind of a problem. It means one guy sees one Gauss's law, and another observer measures a different one. This forces one of three conclusions.

1) You actually see different physical laws in different inertial reference frames. You and I disagree on fields (and the associated forces and accelerations) so completely that one of us will (for example) observe two bodies to repel while the other observes the bodies to attract. No one takes this option very seriously.

2) Maxwell's equations are wrong.

3) Galilean transforms are wrong.

Einstein was pretty fond of Maxwell's equations. They seemed above reproach. It was also known that there were other options for transforms, like the Lorentz transforms:

$$x' = \gamma(x - vt)$$
$$y' = y$$
$$z' = z$$
$$t' = \gamma\left(t - \frac{vx}{c^2}\right)$$

And people knew that if one assumes the speed of light in vacuum is the same in all frames (strongly suggested by Michelson-Morley), one can infer the Lorentz transforms (see homework).

And people knew that Maxwell's equations retained their form under Lorentz transforms. Sounds great, right?
Well, there's a catch. There's always a catch.

$\vec{F} = m\vec{a}$ isn't covariant if you're using Lorentz transforms. So's other stuff. That's bad. That means that if you assert that Lorentz transforms accurately describe our universe, E+M is a-okay, but you're gonna have to rewrite mechanics.

And so Einstein did.