Note Title 10/25/06 10/25/2006

Consider

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L}$$

frequency of terms in F.S. increases with n:

$n=0$ $\quad c_0$

$n=1$ $\quad c_1 e^{i\pi x/L} = e^{i2\pi x/2L}$

freq $= 1/2L$

$n=2$ $\quad c_2 e^{i2\pi x/L} = e^{i2\pi x \frac{2}{2L}}$

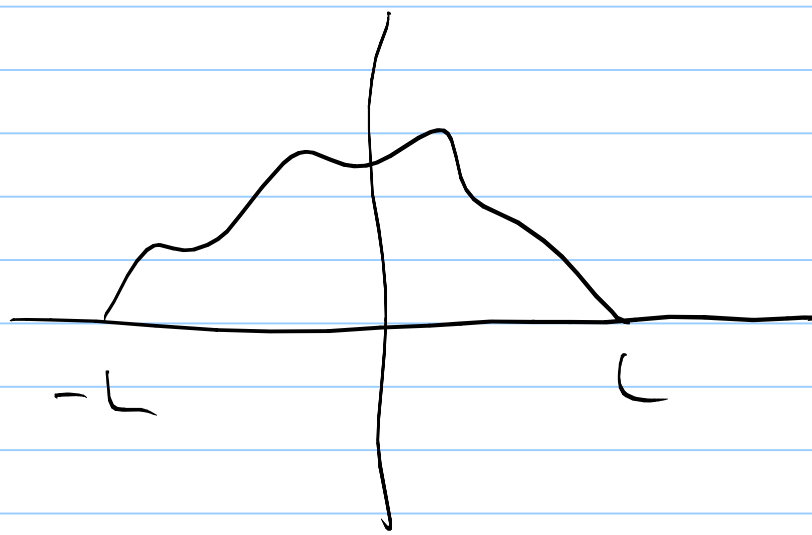
freq $= \frac{2}{2L}$

⋮

$f_1 = \frac{1}{2L}$ $\quad f_2 = \frac{2}{2L}$ $\quad f_3 = \frac{3}{2L}$ $\cdots$

frequency spacing $= \frac{1}{2L}$

Thought experiment



freq. spacing $= \frac{1}{2L}$

let $l = 2L$

frequency spacing?

$$= \frac{1}{2l}$$

frequencies

$$\frac{1}{2l}, \quad \frac{2}{2l}, \quad \frac{3}{2l}, \quad \frac{4}{2l}, \ldots$$

$$\frac{2}{2l} \to \frac{1}{2L} \qquad \frac{4}{2l} \to \frac{2}{2L}$$

half the frequencies are the same.

Even though the F. S. over both intervals have an infinite number of terms, there are twice as many for the interval

$[-2L, 2L] = [-\ell, \ell]$ as

for the interval

$[-L, L]$

on $[-2L, 2L]$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/2L}$$

on $[-L, L]$

$$f(x) = \sum_{n=-\infty}^{\infty} d_n e^{in\pi x/L}$$

twice as many c_n as d_n!

if we keep making the interval larger and larger, the frequency spacing $\to 0$

instead of being a discrete ∞-dimensional vector, c_n becomes a continuous function

$c_n \to F(k)$

finite interval $[-\ell, \ell]$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/\ell}$$

infinite interval

$$f(x) = c_1 \int_{-\infty}^{\infty} F(k)\, e^{ikx}\, dk$$

the coefficien function $F(k)$ is, as usual, determined by orthogonal projection

$$F(k) = c_2 \int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx$$

$$C_1 \times C_2 = \frac{1}{2\pi}$$

So, e.g.

$$\boxed{\begin{aligned} F(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx \\ f(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(k)\, e^{ikx}\, dk \end{aligned}}$$

$F(k)$ + $f(x)$ are Fourier transform pairs

$k = \frac{2\pi}{\lambda}$ $k : x$

$\omega : t$

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega)\, e^{i\omega t}\, d\omega$$

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt$$

or we could do

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)\, e^{i\omega t}\, d\omega$$

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt$$

Example 1

$$f(x) = \begin{cases} 1 & \text{if } -1 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

then $$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} \, dx$$

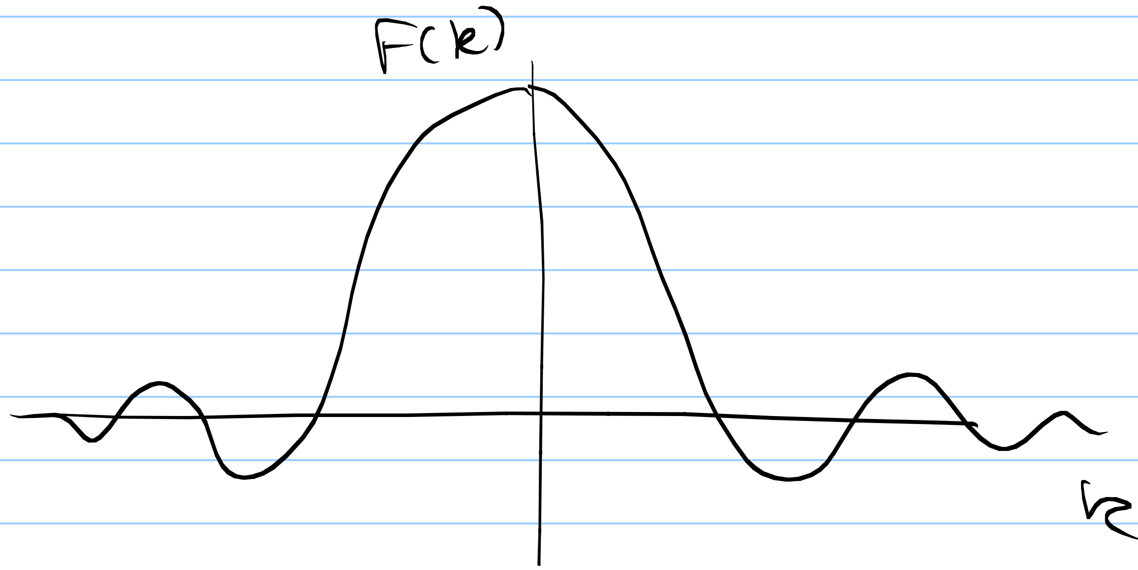
$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-ikx} \, dx$$

$$= \frac{i}{k} \frac{1}{\sqrt{2\pi}} \left[e^{-ikx} \right]_{-1}^{1}$$

$$= \frac{i}{k\sqrt{2\pi}} \underbrace{\left[e^{-ik} - e^{ik} \right]}_{-2i \sin k}$$

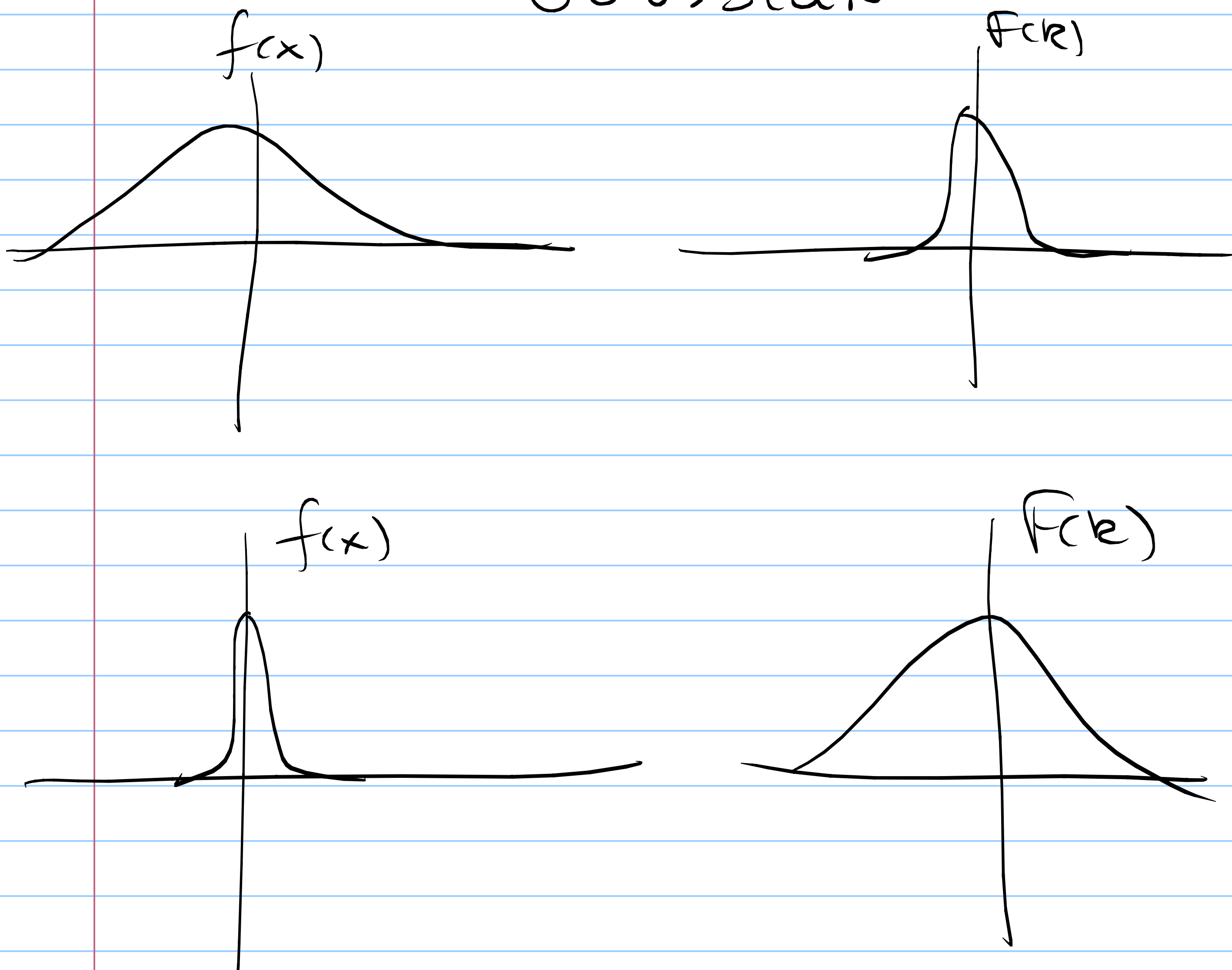
$$= \frac{1}{\sqrt{2\pi}} \; \frac{2\sin k}{k}$$

sinc function



See Mathematica ex.

Gaussian



You need to be able to do the integral

$$I = \int_{-\infty}^{\infty} e^{-x^2}\,dx$$

$$I^2 = \int_{-\infty}^{\infty} e^{-x^2}\,dx \int_{-\infty}^{\infty} e^{-y^2}\,dy$$

$$= \iint_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dx\,dy$$

$$x^2+y^2=r^2$$
$$dx\,dy = r\,dr\,d\theta$$

$$I^2 = \int_0^{\infty}\int_0^{2\pi} e^{-r^2}\, r\, dr\, d\theta$$

$$= 2\pi \cdot \int_0^{\infty} e^{-r^2}\, r\, dr$$

$$r^2 = z \qquad 2r\,dr = dz$$

$$= \frac{2\pi}{2}\int_0^{\infty} e^{-z}\, dz$$

$$\pi \left[-e^{-z} \Big|_0^{\infty} \right]$$

$$I^2 = \pi$$

$$\Rightarrow \quad I = \boxed{\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}}$$