MATH 332 - Linear Algebra June 29, 2009
Homework 4, Summer 2009 **Due**: July 2, 2009

Matrix Equations - Matrix Inversion - Invertible Matrix Theorem - Matrix Partitioning - Matrix Factorization

1. Given,

$$\mathbf{A}(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}, \quad \theta \in (0, 2\pi]. \tag{1}$$

We now consider the *action* of $\mathbf{A}$ on vectors from $\mathbb{R}^2$. That is, we wish to study the effect of the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ represented by the matrix $\mathbf{A}(\theta)$ where $\theta \in (0, 2\pi]$.

   (a) First show that the transformation is one-to-one. [1]
   (b) Given this matrix representation of T find the matrix representation of the inverse transformation. That is find $\mathbf{A}^{-1}$.
   (c) Let $\mathbf{x} = [1\ 0]^{\mathrm{T}}$. Describe or draw the action of the linear transformation $\mathbf{Ax}$ for $\theta \in S$ where $S = \left\{0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}\right\}$. What would the action of $\mathbf{A}^{-1}$ be?
   (d) Let $\mathbf{A}(\theta)\mathbf{x} = \mathbf{b}$ for each $\theta \in S$. Calculate, $\frac{\mathbf{x} \cdot \mathbf{b}}{|\mathbf{x}||\mathbf{b}|}$.[2] How is this related to θ? [3]
   (e) If we define the derivative of a matrix function as a matrix of derivatives then a typical product rule results. That is, if $\mathbf{A}, \mathbf{B}$ have elements, which are functions of θ then $\frac{d\left[\mathbf{AB}\right]}{d\theta} = \mathbf{A}\frac{d\mathbf{B}}{d\theta} + \frac{d\mathbf{A}}{d\theta}\mathbf{B}$. [4] Using this and the identity $\mathbf{AA}^{-1} = \mathbf{I}$ to prove that $\frac{d\left[\mathbf{A}^{-1}\right]}{d\theta} = -\mathbf{A}^{-1}\frac{d\left[\mathbf{A}\right]}{d\theta}\mathbf{A}^{-1}$. Verify this formula using the matrix given above.

2. Given,

$$\mathbf{A} = \begin{bmatrix} 3 & 6 & 7 \\ 0 & 2 & 1 \\ 2 & 3 & 4 \end{bmatrix}.$$

   (a) Calculate $\mathbf{A}^{-1}$ and check your result with the appropriate matrix multiplication.
   (b) Let $\mathbf{A}^{-1}_{\text{left}} = (\mathbf{A}^{\mathrm{T}}\mathbf{A})^{-1}\mathbf{A}^{\mathrm{T}}$. Prove that $\mathbf{A}^{-1}_{\text{left}}$ exists and show that $\mathbf{A}^{-1}_{\text{left}}\mathbf{A} = \mathbf{I}$. [5]
   (c) Let $\mathbf{A}^{-1}_{\text{right}} = \mathbf{A}^{\mathrm{T}}(\mathbf{AA}^{\mathrm{T}})^{-1}$. Prove that $\mathbf{A}^{-1}_{\text{right}}$ exists and show that $\mathbf{AA}^{-1}_{\text{right}} = \mathbf{I}$.[6]
   (d) Let $\mathbf{A}_1 = [2\ 2]^{\mathrm{T}}$ and $\mathbf{A}_2 = [2\ 2]$. Using the previous formula find the left-inverse of $\mathbf{A}_1$ and the right-inverse of $\mathbf{A}_2$. Check your results with the appropriate multiplication.

[1]Recall that a transformation is one-to-one if and only if $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution.

[2]Recall that $\mathbf{x} \cdot \mathbf{y}$ and $|\mathbf{x}|$ are the standard dot-product and magnitude, respectively, from vector-calculus. These operations hold for vectors in $\mathbb{R}^n$ but now have the following definitions, $\mathbf{x} \cdot \mathbf{y} = \mathbf{x}^{\mathrm{T}}\mathbf{y}$ and $|\mathbf{x} = \sqrt{\mathbf{x}^{\mathrm{T}}\mathbf{x}}$.

[3]What we are trying to extract here is the standard result from calculus, which relates the dot-product or inner-product on vectors to the angle between them. This is clear when we have vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$ since we have tools from trigonometry and geometry but when treating vectors in $\mathbb{R}^n, n \geq 4$ these tools are no longer available. However, we would still like to have similar results to those of $\mathbb{R}^n, n = 2, 3$. To make a long story short, we will have these results for arbitrary vectors in $\mathbb{R}^n$ but not immediately. The first thing we must do is show that $|\mathbf{x} \cdot \mathbf{y}| \leq |\mathbf{x}||\mathbf{y}|$, which is known as Schwarz's inequality. Without this we cannot be permitted to always relate $\frac{\mathbf{x} \cdot \mathbf{b}}{|\mathbf{x}||\mathbf{b}|}$ to θ via inverse trigonometric functions. These details will occur in chapter 6 where we find that by using the inner-product on vectors from $\mathbb{R}^n$ we will define the notion of angle and from that distance. Using these definitions and Schwarz's inequality will then give us a triangle-inequality for arbitrary finite-dimensional vectors. This is to say that the algebra of vectors in $\mathbb{R}^n$ carries its own definition of angle and length - very nice of it don't you think? Also, it should be noted that these results exist for certain so-called infinite-dimensional spaces, but are harder to prove - of course, and that the study of linear transformations of such spaces is the general setting for quantum mechanics - see MATH503:Functional Analysis for more details.

[4]To see why this is true differentiate an arbitrary element of $\mathbf{AB}$ to find $\frac{d}{d\theta}\left[\mathbf{AB}\right]_{ij} = \frac{d}{d\theta}\sum_{k=1}^{n} a_{ik}b_{kj} = \sum_{k=1}^{n} \frac{da_{ik}}{d\theta}b_{kj} + a_{ik}\frac{db_{kj}}{d\theta}$.

[5]This matrix is called the left-inverse of $\mathbf{A}$ and it can be shown that if $\mathbf{A} \in \mathbb{R}^{m \times n}$ such that $\mathbf{A}$ has a pivot in every column then the left inverse exists.

[6]This matrix is called the right-inverse of $\mathbf{A}$ and it can be shown that if $\mathbf{A} \in \mathbb{R}^{m \times n}$ such that $\mathbf{A}$ has a pivot in every row then the right inverse exists.

3. Noting any theorems used from class or the text, prove the following statements:

   (a) If $\mathbf{A}$ is an $n \times n$ matrix and $\mathbf{A}^{-1}$ exists, then the columns of $\mathbf{A}$ span $\mathbb{R}^n$.

   (b) If $\mathbf{A}$ is an $n \times n$ matrix and $\mathbf{Ax} = \mathbf{b}$ has a solution for each $\mathbf{b} \in \mathbb{R}^n$, then $\mathbf{A}$ is invertible.

   (c) If the matrix $\mathbf{A}$ is invertible, then the columns of $\mathbf{A}^{-1}$ are linearly independent.

   (d) If the equation $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{A} \in \mathbb{R}^{n \times n}$, has more than one solution for some $\mathbf{b} \in \mathbb{R}^n$, then the columns of $\mathbf{A}$ do not span $\mathbb{R}^n$.

   (e) If the equation $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{A} \in \mathbb{R}^{n \times n}$, is inconsistent for some $\mathbf{b} \in \mathbb{R}^n$, then the equation $\mathbf{Ax} = \mathbf{0}$ has a non-trivial solution.

   (f) If $\mathbf{A}$ is a square matrix with two identical columns then $\mathbf{A}^{-1}$ does not exist.

4. Suppose that $\mathbf{A} \in \mathbb{R}^{n \times n}$ is written in partitioned form as,

$$\mathbf{A} = \begin{bmatrix} \mathbf{P} & \mathbf{Q} \\ \mathbf{R} & \mathbf{S} \end{bmatrix}. \tag{2}$$

   (a) Suppose that $\mathbf{A}$ and $\mathbf{P}$ are non-singular and prove that,

$$\mathbf{A}^{-1} = \begin{bmatrix} \mathbf{X} & -\mathbf{P}^{-1}\mathbf{QW} \\ -\mathbf{WRP}^{-1} & \mathbf{W} \end{bmatrix}, \tag{3}$$

   where $\mathbf{W} = (\mathbf{S} - \mathbf{RP}^{-1}\mathbf{Q})^{-1}$ and $\mathbf{X} = \mathbf{P}^{-1} + \mathbf{P}^{-1}\mathbf{QWRP}^{-1}$.

   (b) Suppose that $\mathbf{A}$ and $\mathbf{S}$ are non-singular and prove that,

$$\mathbf{A}^{-1} = \begin{bmatrix} \mathbf{X} & -\mathbf{XQS}^{-1} \\ -\mathbf{S}^{-1}\mathbf{RX} & \mathbf{W} \end{bmatrix}, \tag{4}$$

   where $\mathbf{X} = (\mathbf{P} - \mathbf{QS}^{-1}\mathbf{R})^{-1}$ and $\mathbf{W} = \mathbf{S}^{-1} + \mathbf{S}^{-1}\mathbf{RXQS}^{-1}$.

   (c) Show that if $\mathbf{P}, \mathbf{S}, \mathbf{A}$ are all non-singular matrices then the two previous forms are equivalent and that $(\mathbf{S} - \mathbf{RP}^{-1}\mathbf{Q})^{-1} = \mathbf{S}^{-1} + \mathbf{S}^{-1}\mathbf{RXQS}^{-1}$.

   (d) Finally, test these previous formula for $\mathbf{P} = a$, $\mathbf{Q} = b$, $\mathbf{R} = c$, $\mathbf{S} = d$ where $a, b, c, d \in \mathbb{R}$ such that $ad - bc \neq 0$.

5. Determine the LU-Decomposition of the matrix $\mathbf{A}$ **and** check your result for $\mathbf{L}$ by multiplication of three elementary matrices.

$$\mathbf{A} = \begin{bmatrix} 1 & 4 & -1 & 5 \\ 3 & 7 & -2 & 9 \\ -2 & -3 & 1 & -4 \end{bmatrix}$$

   Hint: The matrix $\mathbf{U}$, found by three steps of row reduction on $\mathbf{A}$, will have two pivot columns. These two pivot columns are used to determine the first two columns of $\mathbf{L}_{3\times3}$. The remaining column of $\mathbf{L}$ is equal the last column of $\mathbf{I}_3$.