Last Lecture

Final Exam: 4 questions.

1) From first test

2) From second test

3 & 4) Ch.12 relativity.

At least 50% of the test will be from hw and prev. tests.

Conservation Laws

Charge: $\frac{\partial \rho}{\partial t} = -\vec{\nabla}\cdot\vec{J} \Rightarrow \int \frac{\partial \rho}{\partial t} d\tau = -\int \vec{J}\cdot d\vec{a}$

Energy: $\frac{\partial}{\partial t}\left(u_{mech} + u_{em}\right) = -\vec{\nabla}\cdot\vec{S}$

Poynting vector $\vec{S} = \frac{1}{\mu}\vec{E}\times\vec{B}$ Flow of EM energy.

$\frac{\partial}{\partial t} u_{mech}$ = Power density delivered to charges.

$= \vec{F}\cdot\vec{v}$

$= (\rho\vec{E} + \vec{J}\times\vec{B})\cdot\vec{v}$

$= \rho\vec{E}\cdot\vec{v} = \vec{J}\cdot\vec{E}$

$u_{em} = \frac{\epsilon}{2}E^2 + \frac{1}{2\mu}B^2$

$\vec{J}\cdot\vec{E} + \frac{\partial u_{em}}{\partial t} = -\vec{\nabla}\cdot\vec{S}$

Integral form: $\int \vec{J}\cdot\vec{E}\, d\tau + \int \frac{\partial u_{em}}{\partial t} d\tau = -\int \vec{S}\cdot d\vec{a}$

Momentum: momentum dens: momentum per volume

$\vec{P}_{em} = \mu\epsilon\vec{S} = \epsilon_0 \vec{E}\times\vec{B}$

momentum flux dens: flow of momentum per area.

$-\overleftrightarrow{T}: T_{ij} = \epsilon_0\left(E_iE_j - \frac{1}{2}\delta_{ij}E^2\right) + \frac{1}{\mu}\left(B_iB_j - \frac{1}{2}\delta_{ij}B^2\right)$

minus

$\underset{\rho\vec{E}+\vec{J}\times\vec{B}}{\frac{d\vec{P}_{mech}}{dt}} + \underset{\frac{d}{dt}(\epsilon\mu\vec{S})}{\frac{d\vec{P}_{em}}{dt}} = \vec{\nabla}\cdot\overleftrightarrow{T}$

$\int \frac{d\vec{P}_{mec}}{dt} + \frac{d\vec{P}_{em}}{dt} d\tau = \oint \overleftrightarrow{T}\cdot d\vec{a}$

EM Waves

$$\nabla^2\vec{E} - \mu\epsilon\frac{\partial^2\vec{E}}{\partial t^2} = \emptyset \qquad \nabla^2\vec{B} - \mu\epsilon\frac{\partial^2\vec{B}}{\partial t^2} = \emptyset$$

no non-linear ρ_{free}, $\vec{J}_{free}$

potentials

$$\nabla^2 V - \mu_0\epsilon_0\frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\epsilon_0} \quad ; \quad \nabla^2\vec{A} - \mu_0\epsilon_0\frac{\partial^2\vec{A}}{\partial t^2} = -\mu_0\vec{J}$$

Also wave eqns in free space $\rho, \vec{J} = \emptyset$.

In practice, for EM waves, we assumed harmonic t dep.:

$$\vec{E}(\vec{r},t) = \vec{E}(\vec{r})\cos(\omega t + \delta(\vec{r}))$$

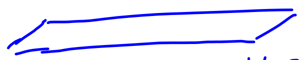
$$= \mathrm{Re}\left[\tilde{\vec{E}}\, e^{-i\omega t}\right]$$

If we do, the wave eqn. becomes:

$$(\nabla^2 + k^2)\vec{E} = \emptyset \quad ; \quad k = \frac{\omega}{c}$$

BCs: (not nec. assuming harmonic t-dep)

material 1: ϵ_1, μ_1



material 2: ϵ_2, μ_2

$$\epsilon_1 E_1^\perp = \epsilon_2 E_2^\perp \qquad \vec{E}_1^{\parallel} = \vec{E}_2^{\parallel}$$

$$B_1^\perp = B_2^\perp \qquad \frac{1}{\mu_1}B_1^{\parallel} = \frac{1}{\mu_2}B_2^{\parallel}$$

Assuming linear materials

True in general

$$\epsilon_1 E_1^\perp - \epsilon_2 E_2^\perp = \sigma_f \qquad \vec{E}_1^{\parallel} - \vec{E}_2^{\parallel} = \emptyset$$

$$B_1^\perp - B_2^\perp = \emptyset \qquad \frac{1}{\mu_1}\vec{B}_1^{\parallel} - \frac{1}{\mu_2}\vec{B}_2^{\parallel} = \vec{K}_f \times \hat{n}$$

surf. current

$\perp$ to surface

Reflection Transmission

$$\tilde{E}_{0R} = \left(\frac{\alpha-\beta}{\alpha+\beta}\right)\tilde{E}_{0I} \qquad \tilde{E}_{0T} = \left(\frac{2}{\alpha+\beta}\right)\tilde{E}_{0I}$$

$$\alpha = \frac{\cos(\theta_T)}{\cos(\theta_I)} \qquad \beta = \frac{\mu_1 n_2}{\mu_2 n_1}$$

$$R = \frac{|\tilde{E}_{0R}|^2}{|\tilde{E}_{0I}|^2} \qquad T = \underbrace{\frac{\epsilon_2 v_2}{\epsilon_1 v_1}}_{\frac{\sqrt{\epsilon_2\mu_1}}{\sqrt{\epsilon_1\mu_2}}} \frac{|\tilde{E}_{0T}|^2}{|\tilde{E}_{0I}|^2}\frac{\cos\theta_T}{\cos\theta_I}$$

$$R = \frac{(\alpha-\beta)^2}{(\alpha+\beta)^2} \qquad T = \alpha\beta\left(\frac{2}{\alpha+\beta}\right)^2$$

Solving waveguide problems

We reduce vector eqn down to

$$\left[\nabla_T^2 + \left(\frac{\omega}{c}\right)^2 - k_z^2\right] E_z = \emptyset$$

$$\text{"} \qquad B_z = \emptyset$$

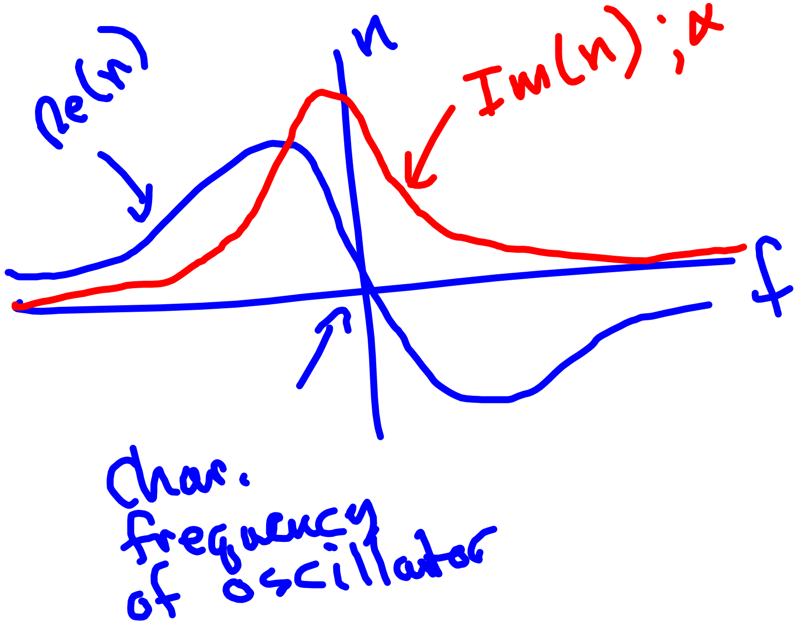
We've assumed $\vec{E}, \vec{B} \sim e^{ik_z z - i\omega t}$

z-direction is down the length of waveguide.

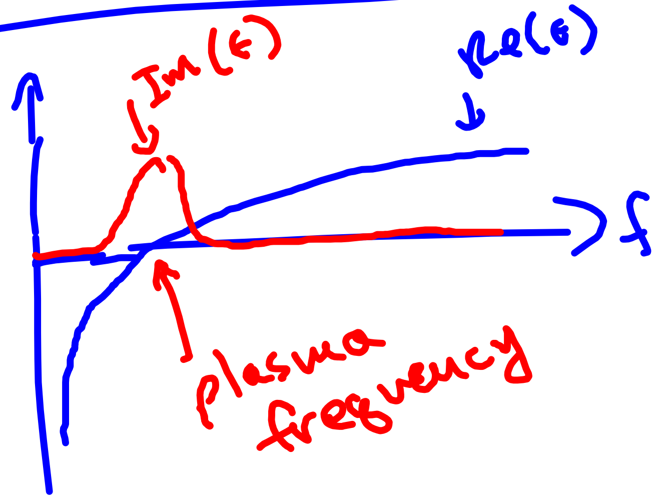
$$\nabla_T^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

Dispersion

Bound electrons



free electrons
Drude Model



Normal dispersion: n increases as f increases

Anomalous " : " decreases " " "

Potentials

$$\vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t} \qquad \vec{B} = \vec{\nabla} \times \vec{A}$$

Gauge transformations:

$$\vec{A}' = \vec{A} + \vec{\nabla}\lambda \quad ; \quad V' = V - \frac{\partial \lambda}{\partial t}$$

for any λ, and this will yield same $\vec{E}, \vec{B}$.

Coloumb gauge: $\vec{\nabla} \cdot \vec{A} = \phi$

Lorentz gauge: $\vec{\nabla} \cdot \vec{A} = -\mu_0 \epsilon_0 \frac{\partial V}{\partial t}$

$$\Box^2 \vec{A} = -\mu_0 J, \quad \Box^2 V = -\rho/\epsilon_0$$

$$\vec{A} = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}', t_r)}{r} d\tau'$$

$$V = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r}', t_r)}{r} d\tau'$$

Fields of a pt. Charge

$$\vec{E}(\vec{r},t) = \frac{q}{4\pi\epsilon_0}\frac{\eta}{(\vec{\eta}\cdot\vec{u})^3}\left[(c^2-v^2)\vec{u} + \vec{\eta}\times(\vec{u}\times\vec{a})\right]$$

$$\vec{u} = c\hat{\eta} - \vec{v}.$$

$$\vec{B}(\vec{r},t) = \frac{1}{c}\vec{\eta}\times\vec{E}(\vec{r},t)$$

Radiation

Power radiated by an elect. dipole

$$P = \frac{\mu_0 \ddot{p}^2}{6\pi c}$$

power

$p = |\vec{p}|$ = electric dipole moment

magnetic dipole

$$P = \frac{\mu_0 \ddot{m}^2}{6\pi c^3}$$

$m = |\vec{m}|$ = magnetic dipole.

Point charge radiation:

non-relativistic

$$P = \frac{\mu_0 q^2 a^2}{6\pi c}$$

relativistic

$$P = \frac{\mu_0 q^2 \gamma^6}{6\pi c}\left(a^2 - \frac{|\vec{v} \times \vec{a}|^2}{c^2}\right)$$

Radiation reaction

$$\vec{F} = \frac{\mu_0 q^2}{6\pi c} \dot{\vec{a}}$$

Special Relativity

Contravariant vec: $\bar{a}^\mu = \frac{\partial \bar{x}^\mu}{\partial x^\nu} a^\nu$

$\uparrow$

Λ^μ_ν

special case of $\bar{S}$ going $v\hat{x}$ w/ resp. to S

$$\Lambda^\mu_\nu = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Covarient vec: $\bar{a}_\mu = \frac{\partial x^\nu}{\partial \bar{x}^\mu} a_\nu$

Invarients: $x_\mu x^\mu = x^2 - c^2t^2$

$\Delta x_\mu \Delta x^\mu = \Delta x^2 - c^2 \Delta t^2$

$$p^\mu = \begin{matrix} \gamma mc \\ \gamma m\vec{u} \end{matrix} = \begin{pmatrix} E/c \\ \vec{p} \end{pmatrix} \Big\} p_\mu p^\mu = p^2 - E^2/c^2 = \text{invarient} = m^2c^2$$

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix}$$

Also $G^{\mu\nu}$ {see lect, book}

Maxwell's eqns:

$$\frac{\partial F^{\mu\nu}}{\partial x^\nu} = \mu_0 J^\mu \quad ; \quad \frac{\partial G^{\mu\nu}}{\partial x^\nu} = \emptyset$$

$$J^\mu = \begin{pmatrix} c\rho \\ \vec{J} \end{pmatrix}$$

For potentials in Lorentz gauge:

$$\partial_\nu \partial^\nu A^\mu = \mu_0 J^\mu$$