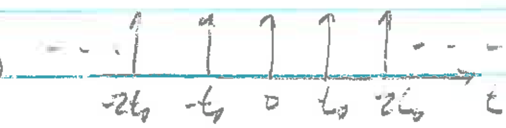
Array theorem

comb function:

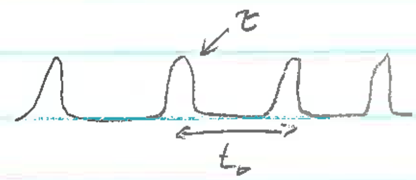
$$\text{comb}(t/t_0) = \sum_{n=-\infty}^{\infty} \delta(t - n t_0)$$



applications:

pulse train:

$$e^{-t^2/\tau^2} \otimes \text{comb}(t/t_0) \rightarrow$$



sampling:

$$g(t) \cdot \text{comb}(t/t_0)$$

t_0 = sampling period.

grating

$$\{a(x) \otimes \text{comb}(x/x_0)\} \cdot \text{rect}(x/D)$$

groove shape   periodicity   D = grating length.

transform:

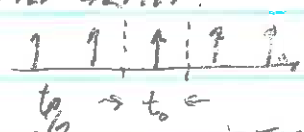
$$f(t) = \sum_{n=-\infty}^{\infty} \delta(t - n t_0)$$

$$F(\omega) = \sum_{n=-\infty}^{\infty} \mathcal{F}\{\delta(t - n t_0)\} = \sum_n e^{-i\omega t_0 n}$$

this is actually a comb function:

comb( ) is periodic, so write as Fourier series:

$$f(t) = \sum_n c_n e^{i 2\pi n t / t_0}$$



$$\text{coeff:} \quad c_n = \frac{1}{t_0} \int_{-t_0/2}^{t_0/2} f(t)\, e^{-i 2\pi n t / t_0}\, dt \Rightarrow \frac{1}{t_0} \int_{-t_0/2}^{t_0/2} \delta(t)\, e^{-i 2\pi n t / t_0}\, dt$$

let limits $\to \pm\infty$

$c_n = 1/t_0$

$\therefore$ we can write

$$\mathrm{comb}(t/t_0) = \frac{1}{t_0}\sum_n e^{i 2\pi n t/t_0}$$

$$\mathcal{F}\{\mathrm{comb}(t/t_0)\} = \frac{1}{t_0}\sum_n \mathcal{F}\{e^{i\frac{2\pi n}{t_0}t}\}$$

$$= \sum_n \frac{2\pi}{t_0}\delta\left(\omega + \frac{2\pi n}{t_0}\right)$$

$$= \frac{2\pi}{t_0}\mathrm{comb}\left(\frac{\omega}{(2\pi/t_0)}\right)$$

frequency spacing $\Delta\omega = \frac{2\pi}{t_0} \to \Delta\nu = \frac{1}{t_0}$

in a laser resonator: $t_0 = T_{RT} = \frac{2L}{c}$ round trip time.

resonator

to have $E(0) = E(L) = 0$

$\to E \propto \sin\left(\frac{m\pi}{L}x\right) \propto \left(e^{\frac{m\pi}{L}x} - e^{-\frac{m\pi}{L}x}\right)$ $\quad k_m = \frac{m\pi}{L}$

allowed frequencies $\omega_m = k_m c = \frac{m\pi c}{L}$ $\quad \nu_m = m \cdot \frac{c}{2L} = \frac{m}{T_{RT}}$

T_{RT} = round trip time

if all allowed frequencies are present, then

$$E(t) \propto \sum_m e^{-i\omega_m t} = \sum_m e^{-i\frac{2\pi m}{T_{RT}}t} \propto \sum_m \delta(t - nT_{RT})$$

$$\propto \mathrm{comb}(t/T_{RT})$$

this represents a sequence of δ-fcn pulses separated by T_{RT}

spectrum $E(\omega) \propto \mathrm{comb}(-\omega/\delta\omega)$ $\quad \delta\omega = 2\pi/T_{RT}$

= "frequency comb"

Mode-locked pulse train

Start in frequency space:

assume a Gaussian gain bandwidth $e^{-(\omega-\omega_0)^2/\Delta\omega_L^2}$

cavity $\rightarrow$ $\mathrm{comb}(\omega/\Delta\omega_c)$ (ideal)

spectrum: $F(\omega) = \left(e^{-(\omega-\omega_0)^2/\Delta\omega_L^2}\right) \cdot \mathrm{comb}(\omega/\Delta\omega_c)$

$$= G(\omega) \cdot H(\omega)$$

in time domain:

$$f(t) = g(t) \otimes h(t)$$

$$g(t) = \mathcal{F}^{-1}\left\{e^{-(\omega-\omega_0)^2/\Delta\omega_L^2}\right\} = e^{-i\omega_0 t}\,\mathcal{F}\left\{e^{-\omega^2/\Delta\omega_L^2}\right\}$$

$$= \frac{1}{\sqrt{\pi t_p^2}}\, e^{-i\omega_0 t}\, e^{-t^2/t_p^2} \qquad t_p = 2/\Delta\omega_L$$

$$h(t) = \mathcal{F}^{-1}\left\{\mathrm{comb}(\omega/\Delta\omega_c)\right\}$$

since $\mathcal{F}\{\mathrm{comb}(t/t_0)\} = \left(\frac{2\pi}{t_0}\right)\mathrm{comb}\left(\omega/(2\pi/t_0)\right)$

let $\Delta\omega_c = 2\pi/t_0$ $\quad \Delta\nu_c = 1/t_0$ $\quad t_0$ = roundtrip time

$$\rightarrow h(t) = \frac{1}{\Delta\omega_c}\,\mathrm{comb}(t/t_0)$$

$$f(t) = g(t) \otimes h(t) = \frac{1}{\sqrt{4\pi}}\,\frac{\Delta\omega_L}{\Delta\omega_c}\left(e^{-i\omega_0 t}\, e^{-t^2/t_p^2} \otimes \mathrm{comb}(t/t_0)\right)$$

Sampling + signal recovery

- ubiquitous: digitized signals, numerical analysis

sampled function

$$f_s(t) = f(t)\,\mathrm{comb}(t/t_s)$$

multiply, not convolve.

$$= \sum_n f(t)\,\delta(t - n t_s)$$

spectrum: (use convolution theorem)

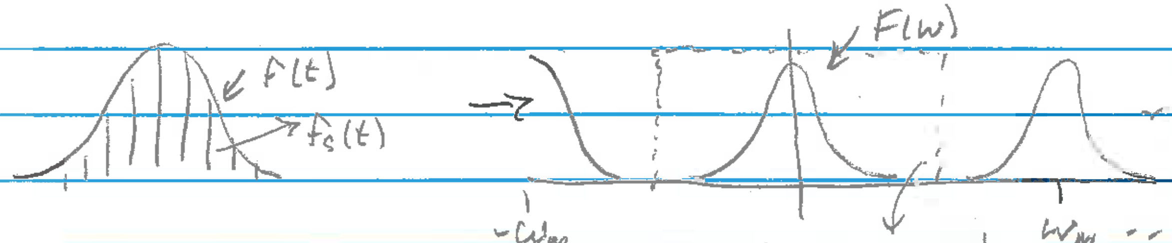
$$F_s(\omega) = \frac{1}{2\pi} F(\omega) \otimes \frac{2\pi}{t_s}\,\mathrm{comb}\left(\frac{\omega}{2\pi/t_s}\right)$$

$$= \frac{1}{t_s} \sum_n F\left(\omega - n\frac{2\pi}{t_s}\right)$$

↳ $\equiv \omega_M$ = max span of ω

$F_s(\omega)$ is a continuous function

replicas of $F(\omega)$ spaced by ω_M



$H(\omega) = \mathrm{rect}(\omega/\omega_M)$

if there is no overlap in spectra:

- low-pass filter, recover exact original $F(\omega) \to f(t)$
- input signal must be BW limited for no overlap
- if full width of input spectrum is W
  - ↳ edge to edge
- sampling rate $\omega_M \geq W$ $\quad t_s \leq \frac{2\pi}{W}$

or max freq. is $W/2 = \omega_{max} \to$ period $T_{min} = \frac{2\pi}{\omega_{max}}$

$t_s \leq \frac{1}{2} T_{min}$ Nyquist limit note that many filters are characterized by their $\frac{1}{2}$ power width

$\therefore$ higher frequencies get through.

$\rightarrow$ "oversampling" is required to make sure $F(\omega) \to 0$ at boundaries.

signal recovery: use low-pass filter

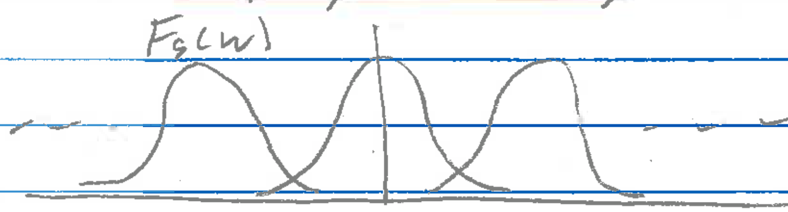
$$G(\omega) = H(\omega) F_s(\omega)$$
$$= \mathrm{rect}\left[\omega/\omega_s\right] \cdot \frac{1}{t_s} \sum_n F(\omega - n\omega_s)$$

alternatively,

$$g(t) = h(t) \otimes f_s(t)$$
$$= \frac{\omega_s}{2\pi} \mathrm{sinc}\left(\frac{\omega_s}{2} t\right) \otimes f_s(t)$$

so the sinc( ) acts as the perfect interpolating function.

Undersampling $\rightarrow$ aliasing



high ω wraps around $\rightarrow$ looks like lower ω.

Limited sampling duration:

- clips input

$$f_s'(t) = f(t) \cdot \mathrm{comb}(t/t_s) \cdot \mathrm{rect}(t/T)$$

$$F_s'(\omega) = \frac{1}{2\pi} F_s(\omega) \otimes T\, \mathrm{sinc}(\omega T/2)$$

☆ spectral resolution is smeared out by finite T