MATH332 - July 8, 2009

Exam I - 50 Points

12:47pm

NAME: Key

In order to receive full credit, SHOW ALL YOUR WORK. Full credit will be given only if all reasoning and work is provided. When applicable, please enclose your final answers in boxes.

1. (10 Points) True/False and Short Response

   (a) Mark each statement True or False.

      i. If $\mathbf{A}$ is an invertible matrix then $\mathbf{A}^{-1}\mathbf{x} = \mathbf{0}$ has a non-trivial solution.

      False

      ii. Any set of n vectors from $\mathbb{R}^m$, such that $n > m$, forms a linearly dependent set.

      True

      iii. If $\mathbf{A}_{m\times n}$ has a pivot in every column then its columns span $\mathbb{R}^m$.

      False

      iv. If $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n\times n}$ then $\det(\mathbf{A}+\mathbf{B}) = \det(\mathbf{A}) + \det(\mathbf{B})$

      False

      v. The determinant of an elementary matrix corresponding to a row-interchange is 1.

      False

   (b) Please respond to **one** of the following questions and justify your position:

      i. Suppose $\mathbf{A}$ is a 4×3 matrix with the property that $\mathbf{Ax} = \mathbf{0}$ has a unique solution. What can yo say about the reduced echelon form of $\mathbf{A}$? What can you conclude about the system $\mathbf{Ax} = \mathbf{b}$ wher $\mathbf{b}$ is a vector in $\mathbb{R}^4$?

      ii. Suppose $\mathbf{A}$ is a 3×3 matrix and $\mathbf{y}$ is a vector in $\mathbb{R}^3$ such that the equation $\mathbf{Ax} = \mathbf{y}$ does <u>not</u> hav a solution. Does there exist a vector $\mathbf{z}$ in $\mathbb{R}^3$ such that the equation $\mathbf{Ax} = \mathbf{z}$ has a unique solution

ii) If $A\vec{x}=\vec{y}$ has no sol$\underline{n}$ $\Rightarrow$ A is not invertible $\Rightarrow$ Free vars $\Rightarrow$ $A\vec{x}=\vec{z}$ has no sol$\underline{n}$ or non unique sol$\underline{n}$

i) $A\vec{x}=\vec{0}$ only $\vec{x}=\vec{0}$ $\Rightarrow$ 3 pivots $\Rightarrow A \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow A\vec{x}=\vec{b}$ May have a unique Sol$\underline{n}$ or no sol$\underline{n}$.

2. (10 Points) Quickies: $\left[\begin{array}{cc|c} 1 & 3 & 2 \\ 3 & h & k \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 3 & 3 \\ 0 & h-3 & k-2 \end{array}\right]$

(a) Let $\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 3 & h \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 2 \\ k \end{bmatrix}$. Find **all** values of h and k for which the system $\mathbf{Ax} = \mathbf{b}$ is:

i. Consistent with a unique solution.

$h \neq 3,\ k \in \mathbb{R}$

ii. Consistent with infinity-many solutions.

$h = 3,\ k = 2$

iii. Inconsistent.

$h = 3,\ k \neq 2$

(b) Invert the following matrices,

$$\mathbf{A}_1 = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}, \quad \mathbf{A}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad \mathbf{A}_4 = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}.$$

$A_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $A_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$, $A_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$, $A_4 = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$

(c) ~~Solve the following linear system and~~ Describe the geometry of the general solution set, to

$$\begin{aligned} 3x_1 - 9x_2 + 6x_3 &= 0 \quad (1 \\ -x_1 + 3x_2 - 2x_3 &= 0 \quad (2 \end{aligned}$$

$\left[\begin{array}{ccc|c} 3 & -9 & 6 & 0 \\ -1 & 3 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 3 & -9 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ There are the same Plane!

3. (10 Points) Proofs:

(a) If $\mathbf{A}$ is invertible prove that $\mathbf{B} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T$ exists and that $\mathbf{B} = \mathbf{A}^{-1}$.

A^{-1} Exists $\Rightarrow (A^T)^{-1}$ Exists $\Rightarrow (A^TA)^{-1}$ Exists $\Rightarrow$ B Exists

$B = (A^TA)^{-1}A^T = A^{-1}(A^T)^{-1}A^T = A^{-1}$ ✓

(b) Prove that $\mathbf{A}\mathbf{A}^T$ is a symmetric matrix.

$(AA^T)^T = A^{T^T}A^T = AA^T$ ✓

(c) Prove that $(\mathbf{A} + \mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$

$[(A+B)^T]_{ij} = [A+B]_{ji} = a_{ji} + b_{ji}$

$[A^T + B^T]_{ij} = a_{ji} + b_{ji} \Rightarrow (A+B)^T = A^T + B^T$

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4. (10 Points) Given,

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -3 \\ 9 \\ -6 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 5 \\ -7 \\ h \end{bmatrix}.$$

(a) For what values of h is $\mathbf{v}_3$ in the Span$\{\mathbf{v}_1, \mathbf{v}_2\}$?

(b) For what values of h are the vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ a linearly dependent?

a) $$\left[\begin{array}{ccc|c} 1 & -3 & 5 & 0 \\ -3 & 9 & -7 & 0 \\ 2 & -6 & h & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -3 & 5 & 0 \\ 0 & 0 & 8 & 0 \\ 0 & 0 & h-10 & 0 \end{array}\right]$$

$V_3 \notin \text{span}\{\vec{v}_1, \vec{v}_2\}$ for any h!

b) $\vec{v}_1, \vec{v}_2$ are lin. dep. $\Rightarrow \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is always lin. dep.

5. (10 Points) Given,

$$\mathbf{A} = \begin{bmatrix} 3 & 6 & 7 \\ 0 & 2 & 1 \\ 2 & 3 & 4 \end{bmatrix}, \quad \mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2/3 & -1/2 & 1 \end{bmatrix}.$$

(a) Find **U** associated with the **LU** decomposition of **A**.

Hint: Don't forget what information is encoded into **L**!

$$A = \begin{bmatrix} 3 & 6 & 7 \\ 0 & 2 & 1 \\ 2 & 3 & 4 \end{bmatrix} \underset{R3 = R3 - \frac{2}{3}R1}{\sim} \begin{bmatrix} 3 & 6 & 7 \\ 0 & 2 & 1 \\ 0 & -1 & -2/3 \end{bmatrix} \underset{R2 = R2 + \frac{1}{2}R1}{\sim} \begin{bmatrix} 3 & 6 & 7 \\ 0 & 2 & 1 \\ 0 & 0 & -1/6 \end{bmatrix} = U$$

$3 - \frac{2}{3} \cdot 6 =$

$4 - \frac{2}{3} \cdot 7 = \frac{12}{3} - \frac{14}{3} = -\frac{2}{3}$

$-\frac{2}{3} + \frac{1}{2} = \frac{-4+3}{6}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{2}{3} & -\frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 3 & 6 & 7 \\ 0 & 2 & 1 \\ 0 & 0 & -\frac{1}{6} \end{bmatrix} = \begin{bmatrix} 3 & 6 & 7 \\ 0 & 2 & 1 \\ 2 & 3 & 4 \end{bmatrix}$$

$\frac{2}{3} \cdot 6 - \frac{1}{2} \cdot 2 = 4 - 1 = 3, \quad \frac{2}{3} \cdot 7 - \frac{1}{2} - \frac{1}{6} = \frac{14}{3} - \frac{1}{2} - \frac{1}{6} = \frac{28}{6} - \frac{3}{6} - \frac{1}{6} = \frac{24}{6} = 4$

(b) Using the **LU** decomposition solve $\mathbf{Ax} = \mathbf{b}$ where $\mathbf{b} = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$.

$LUx = b \Leftrightarrow L\vec{y} = b, \; U\vec{x} = \vec{y}$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 2/3 & -1/2 & 1 & -1 \end{array}\right] \Rightarrow \begin{matrix} y_1 = 0 \\ y_2 = 2 \\ y_3 = 0 \end{matrix} \Rightarrow \left[\begin{array}{ccc|c} 3 & 6 & 7 & 0 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & -1/6 & 0 \end{array}\right] \Rightarrow \begin{matrix} x_1 = -2 \\ x_2 = 1 \\ x_3 = 0 \end{matrix} \Rightarrow \vec{x} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

(c) Calculate the determinant of **A**.

Hint: Properties of determinants should make this calculation very straightforward.

$\det(A) = \det(LU) = \det(L)\det(U) = 1 \cdot 3 \cdot 2 \cdot -\frac{1}{6} = -1$

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6. (Extra Credit 1) True/False,

   (a) If the equation $\mathbf{Ax} = \mathbf{b}$ is inconsistent, then $\mathbf{b}$ is not in the set spanned by the columns of $\mathbf{A}$.

   True.

   (b) If $\mathbf{x}$ is a non-trivial solution to $\mathbf{Ax} = \mathbf{0}$, then every entry in $\mathbf{x}$ is non-zero.

   False

   (c) If $\mathbf{x}$ and $\mathbf{y}$ are linearly independent and if $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ is a linearly dependent set then $\mathbf{z}$ is in Span$\{\mathbf{x}, \mathbf{y}\}$

   True

   (d) If $\mathbf{A}$ and $\mathbf{B}$ are diagonal matrices then $\mathbf{AB}=\mathbf{BA}$.

   True

   (e) $\det(\mathbf{A}^T) = -\det(\mathbf{A})$

   False

7. (Extra Credit 2) Invert the matrix from problem 5. Check your work with the appropriate matrix multipli cation.

   See HW # ~~3~~4.