Homework #2 Solution:

1. We have the polynomial

$$p(t) = a_0 + a_1 t + a_2 t^2$$

and the data points $(1,12),(2,15),(3,16)^{(\star)}$. This generates 3 linear equations

$$\begin{aligned} p(1) &= a_0 + a_1 + a_2 = 12 \\ p(2) &= a_0 + 2a_1 + 4a_2 = 15 \\ p(3) &= a_0 + 3a_1 + 9a_2 = 16 \end{aligned}$$

and the corresponding augmented matrix.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 12 \\ 1 & 2 & 4 & 15 \\ 1 & 3 & 9 & 16 \end{array}\right] \sim_{\substack{R3=R3-R1 \\ R2=R2-R1}} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 12 \\ 0 & 1 & 3 & 3 \\ 0 & 2 & 8 & 4 \end{array}\right] \sim$$

$$\sim_{R3=R3-2R2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 12 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 2 & -2 \end{array}\right] \sim_{R3=R3/2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 12 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right] \sim$$

$$\sim_{\substack{R1=R1-R3 \\ R2=R2-3R3}} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 13 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & -1 \end{array}\right] \sim_{R1=R1-R2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & -1 \end{array}\right] \sim$$

The row equivalent linear system is then,

$$\begin{aligned} a_0 &= 7 \\ a_1 &= 6 \\ a_2 &= -1 \end{aligned}$$

which implies that $p(t) = 7 + 6t - t^2$ is the quadratic polynomial which indicates $(\star)$.

2. If $\vec{x} = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$ then the new vectors $A(\theta)\vec{x}$ will correspond to the vectors on the unit circle. A represents ridged (norm-preserving) rotations (counter clockwise) of $\vec{x}$. $A = \begin{bmatrix} cos(\theta) & sin(\theta) \\ -sin(\theta) & cos(\theta) \end{bmatrix}$ represents clockwise rotations.

3. $A = \begin{bmatrix} 3 & 6 & 7 \\ 0 & 2 & 1 \\ 2 & 3 & 4 \end{bmatrix}$ Determine A^{-1} via

a) calculate $det(A)$.

$$\begin{aligned} det(A) &= 3det\begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix} - 0 \cdot det\begin{pmatrix} 6 & 7 \\ 3 & 4 \end{pmatrix} + 2det\begin{pmatrix} 6 & 7 \\ 2 & 1 \end{pmatrix} \\ &= 3(5) - 0(3) + 2(-8) = 15 \cdot 16 = -1 \end{aligned}$$

b) The Gauss-Jordan Method

$$\left[\begin{array}{ccc|ccc} 3 & 6 & 7 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 2 & 3 & 4 & 0 & 0 & 1 \end{array}\right] \begin{array}{c} R1-3R2 \\ \\ 2R1-3R3 \end{array} \left[\begin{array}{ccc|ccc} 3 & 0 & 4 & 1 & -3 & 0 \\ 0 & 2 & 1 & & 0 & 10 \\ 0 & 3 & 2 & 2 & 0 & -3 \end{array}\right] \; 2R3-3R2$$

$$\sim \left[\begin{array}{ccc|ccc} 3 & 0 & 4 & 1 & -3 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 4 & -3 & -6 \end{array}\right] \begin{array}{c} R1-4R3 \\ R2-R3 \\ . \end{array} \left[\begin{array}{ccc|ccc} 3 & 0 & 0 & -15 & -9 & 24 \\ 0 & 2 & 0 & -4 & 4 & 6 \\ 0 & 0 & 1 & 4 & -3 & -6 \end{array}\right] \begin{array}{c} \div 3 \\ \div 2 \\ . \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -5 & 3 & 8 \\ 0 & 1 & 0 & -2 & 2 & 3 \\ 0 & 0 & 1 & 4 & -3 & -6 \end{array}\right] \qquad A^{-1} = \begin{bmatrix} -5 & 3 & 8 \\ -2 & 2 & 3 \\ 4 & -3 & -6 \end{bmatrix}$$

c) The Cofactor Representation

$$A^{-1} = \frac{1}{det(A)} \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} = \frac{1}{-1} \begin{bmatrix} 5 & -3 & 8 \\ 2 & -2 & -3 \\ -4 & 3 & 6 \end{bmatrix} = \begin{bmatrix} -5 & 3 & 8 \\ -2 & 2 & 3 \\ 4 & -3 & -6 \end{bmatrix}$$

d) Check your result by showing $AA^{-1} = I$

$$AA^{-1} = \begin{bmatrix} 3 & 6 & 7 \\ 0 & 3 & 1 \\ 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} -5 & 3 & 8 \\ -2 & 2 & 3 \\ 4 & -3 & -6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

4.

i. $det(A) = ad - bc$

ii. $det(B) = cb - ad = -(ad - bc) = -det(A)$

iii. $det(D) = d(a + kc) - c(b + kd) = ad + kdc - cb - cdk = ad - bc = det(A)$

iv. $det(C) = adk - kcb = k(ad - bc) = k \cdot det(A)$

$A \sim B$ by a row interchange and ii shows $det(A) = -det(B)$
$A \sim C$ by a row scaling and iv shows $det(A) = k \cdot det(C)$
$A \sim D$ by a row interchange where a multiple of one row is added to another.
iii shows that $det(A) = det(D)$

5.
Forward Direction: Assume $A_{3\times x}$ is such that $det(A) = 0$. then the volume of the parallelopiped spaned by $\vec{a}_1, \vec{a}_2, \vec{a}_3$ has zero volume. That is, the parallelopiped does not exist. This implies that all the vectors $\vec{a}_1, \vec{a}_2, \vec{a}_3$ lie in the same plane, and from a linearly dependent set. Thus, by the invertable matrix theorem A^{-1} does not exist.

Backward Direction: Assume A is not invertable. Then the columns of A are linearly dependent and $span\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$ is at most (in terms of dimension) a plane which has zero volume and cannot form a parallelopiped. Thus, $det(A) = 0$.