Problem 4.9

(a) $\mathbf{F} = (\mathbf{p}\cdot\boldsymbol{\nabla})\mathbf{E}$ (Eq. 4.5); $\mathbf{E} = \dfrac{1}{4\pi\epsilon_0}\dfrac{q}{r^2}\,\hat{\mathbf{r}} = \dfrac{q}{4\pi\epsilon_0}\dfrac{x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}}{(x^2+y^2+z^2)^{3/2}}$.

$$
\begin{aligned}
F_x &= \left(p_x\frac{\partial}{\partial x} + p_y\frac{\partial}{\partial y} + p_z\frac{\partial}{\partial z}\right)\frac{q}{4\pi\epsilon_0}\frac{x}{(x^2+y^2+z^2)^{3/2}} \\
&= \frac{q}{4\pi\epsilon_0}\left\{p_x\left[\frac{1}{(x^2+y^2+z^2)^{3/2}} - \frac{3}{2}x\frac{2x}{(x^2+y^2+z^2)^{5/2}}\right] + p_y\left[-\frac{3}{2}x\frac{2y}{(x^2+y^2+z^2)^{5/2}}\right]\right. \\
&\quad \left. + p_z\left[-\frac{3}{2}x\frac{2z}{(x^2+y^2+z^2)^{5/2}}\right]\right\} = \frac{q}{4\pi\epsilon_0}\left[\frac{p_x}{r^3} - \frac{3x}{r^5}(p_x x + p_y y + p_z z)\right] = \frac{q}{4\pi\epsilon_0}\left[\frac{\mathbf{p}}{r^3} - \frac{3\mathbf{r}(\mathbf{p}\cdot\mathbf{r})}{r^5}\right]_x. \\
\mathbf{F} &= \boxed{\frac{1}{4\pi\epsilon_0}\frac{q}{r^3}\left[\mathbf{p} - 3(\mathbf{p}\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}}\right].}
\end{aligned}
$$

(b) $\mathbf{E} = \dfrac{1}{4\pi\epsilon_0}\dfrac{1}{r^3}\{3\,[\mathbf{p}\cdot(-\hat{\mathbf{r}})]\,(-\hat{\mathbf{r}}) - \mathbf{p}\} = \dfrac{1}{4\pi\epsilon_0}\dfrac{1}{r^3}\left[3(\mathbf{p}\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}} - \mathbf{p}\right]$. (This is from Eq. 3.104; the minus signs are because $\mathbf{r}$ points *toward* $\mathbf{p}$, in this problem.)

$$\mathbf{F} = q\mathbf{E} = \boxed{\frac{1}{4\pi\epsilon_0}\frac{q}{r^3}\left[3(\mathbf{p}\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}} - \mathbf{p}\right].}$$

[Note that the forces are equal and opposite, as you would expect from Newton's third law.]

Problem 4.21

Let Q be the charge on a length ℓ of the inner conductor.

$$
\begin{aligned}
\oint \mathbf{D}\cdot d\mathbf{a} &= D2\pi s\ell = Q \Rightarrow D = \frac{Q}{2\pi s\ell}; \quad E = \frac{Q}{2\pi\epsilon_0 s\ell}\ (a < s < b), \quad E = \frac{Q}{2\pi\epsilon s\ell}\ (b < r < c). \\
V &= -\int_c^a \mathbf{E}\cdot d\mathbf{l} = \int_a^b\left(\frac{Q}{2\pi\epsilon_0\ell}\right)\frac{ds}{s} + \int_b^c\left(\frac{Q}{2\pi\epsilon\ell}\right)\frac{ds}{s} = \frac{Q}{2\pi\epsilon_0\ell}\left[\ln\left(\frac{b}{a}\right) + \frac{\epsilon_0}{\epsilon}\ln\left(\frac{c}{b}\right)\right]. \\
\frac{C}{\ell} &= \frac{Q}{V\ell} = \boxed{\frac{2\pi\epsilon_0}{\ln(b/a) + (1/\epsilon_r)\ln(c/b)}}.
\end{aligned}
$$

Problem 4.33

$\mathbf{E}^{\parallel}$ is continuous (Eq. 4.29); $D_\perp$ is continuous (Eq. 4.26, with $\sigma_f = 0$). So $E_{x_1} = E_{x_2}$, $D_{y_1} = D_{y_2} \Rightarrow \epsilon_1 E_{y_1} = \epsilon_2 E_{y_2}$, and hence

$$\frac{\tan\theta_2}{\tan\theta_1} = \frac{E_{x_2}/E_{y_2}}{E_{x_1}/E_{y_1}} = \frac{E_{y_1}}{E_{y_2}} = \frac{\epsilon_2}{\epsilon_1}. \quad \text{qed}$$

If 1 is air and 2 is dielectric, $\tan\theta_2/\tan\theta_1 = \epsilon_2/\epsilon_0 > 1$, and the field lines bend *away* from the normal. This is the opposite of light rays, so a convex "lens" would *defocus* the field lines.

Problem 1.57

$\mathbf{v}\cdot d\mathbf{l} = y\,dz$.

(1) *Left side:* $z = a - x$; $dz = -dx$; $y = 0$. Therefore $\int \mathbf{v}\cdot d\mathbf{l} = 0$.

(2) *Bottom:* $dz = 0$. Therefore $\int \mathbf{v}\cdot d\mathbf{l} = 0$.

(3) *Back:* $z = a - \frac{1}{2}y$; $dz = -1/2\,dy$; $y : 2a \to 0$. $\int \mathbf{v}\cdot d\mathbf{l} = \int_{2a}^{0} y\left(-\frac{1}{2}\,dy\right) = -\frac{1}{2}\frac{y^2}{2}\Big|_{2a}^{0} = \frac{4a^2}{4} = \boxed{a^2.}$

Meanwhile, $\boldsymbol{\nabla}\times\mathbf{v} = \hat{\mathbf{x}}$, so $\int(\boldsymbol{\nabla}\times\mathbf{v})\cdot d\mathbf{a}$ is the projection of this surface on the xy plane $= \frac{1}{2}\cdot a\cdot 2a = a^2$. ✓