Today: Newtonian → Lagrangian (7.6); 7.1 → 7.3
Monday: Lagrangian (more); 7.4 → 7.7.

Hamilton's Principle: Nature minimizes:

$$\int_{t_1}^{t_2} \underset{\uparrow \text{ Kin E}}{T} - \underset{\nwarrow \text{ Pot. E}}{U} \, dt \qquad \underset{\uparrow \text{ Lagrangian}}{L} = T - U$$

$$\Rightarrow \quad \frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \phi \qquad \text{in 1-D.}$$

Let's derive it from Newton's 2^{nd} Law:

$$\underbrace{\Sigma \vec{F}}_{\text{split it into}} = \dot{\vec{p}}$$

split it into $\Sigma \vec{F}_a + \Sigma \vec{F}_c$ ← constraints.

$$\Sigma \vec{F}_a + \Sigma \vec{F}_c - \dot{\vec{p}} = \phi$$

If our object were displaced by a "virtual displacement" $\delta \vec{r}$, not a real displacement, an arbitrary displacement I control.

$$\left(\sum \vec{F}_a + \sum \vec{F}_c - \dot{\vec{p}}\right) \cdot \delta \vec{r} = \emptyset$$

Assumption: $\sum \vec{F}_c \cdot \delta \vec{r}$ for all reasonable $\delta \vec{r}$ is $\emptyset$.

$$\sum \vec{F}_a \cdot \delta \vec{r} - \dot{\vec{p}} \cdot \delta \vec{r} = \emptyset \; : \; \left.\begin{aligned} F_{ax}\,\delta x - \dot{p}_x\,\delta x &= \emptyset \\ F_{ay}\,\delta y - \dot{p}_y\,\delta y &= \emptyset \\ F_{az}\,\delta z - \dot{p}_z\,\delta z &= \emptyset \end{aligned}\right\}$$

No sums Remember we mean net force.

or $\sum_i \left(F_{ai}\,\delta x_i - \dot{p}_i\,\delta x_i \right) = \emptyset$; $i = 1, 2, 3.$

If I want to change coordinates:

e.g. spherical polar coord. $x = x(r, \theta, \phi)$; $y = y(r, \theta, \phi)$; $z = z(r, \theta, \phi)$
$= r\sin\theta\cos\phi$ $\quad = r\sin\theta\sin\phi$ $\quad = r\cos\theta$

2-D polar coord : $x = r\cos\theta$; $y\ r\sin\theta$

$$x_i = x_i(q_1, q_2, q_3 ; t) = x_i(q_j ; t)$$

$$\begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ r & \theta & \phi \end{array}$$

$$\dot{x}_i = \sum_j \frac{\partial x_i}{\partial q_j}\dot{q}_j + \frac{\partial x_i}{\partial t} \quad ; \text{ tan.} ☺$$

$$\delta x_i = \sum_j \frac{\partial x_i}{\partial q_j}\delta q_j$$

$$\Rightarrow \sum_i F_{ai}\,\delta x_i = \sum_{i,j} \underbrace{F_{ai}\frac{\partial x_i}{\partial q_j}}_{Q_j}\delta q_j = \sum_j Q\,\delta q_j$$

$Q_j \equiv$ generalized force in the q_j direction e.g. 2D polar: $\sum F_{ai}\delta x_i = F_{ax}\delta x + F_{ay}\delta y$ $\quad x = r\cos\theta$, $y = r\sin\theta$

$$\delta x_i = \sum_j \frac{\partial x_i}{\partial q_j}\delta q_j$$

$$\delta x = \frac{\partial x}{\partial r}\delta r + \frac{\partial x}{\partial \theta}\delta\theta \;;\; \delta y = \frac{\partial y}{\partial r}\delta r + \frac{\partial y}{\partial \theta}\delta\theta$$

$$\delta x = \cos\theta\,\delta r - r\sin\theta\,\delta\theta \;;\; \delta y = \sin\theta\,\delta r + r\cos\theta\,\delta\theta$$

$$\sum_i F_{ai}\delta x_i = \sum_j Q_j \delta q_j$$

$$= F_{ax}(\cos\theta\,\delta r - r\sin\theta\,\delta\theta) + F_{ay}(\sin\theta\,\delta r + r\cos\theta\,\delta\theta)$$

$$= \underbrace{(F_{ax}\cos\theta + F_{ay}\sin\theta)}_{Q_r = F_{ar}}\delta r + \underbrace{(-rF_{ax}\sin\theta + rF_{ay}\cos\theta)}_{Q_\theta = \tau_z}\delta\theta$$

N·m

Back to $\sum_j Q_j \delta q_j - \sum_i m\ddot{x}_i \delta x_i = \emptyset$

Consider: $\frac{d}{dt}\left(m\dot{x}_i \frac{\partial x_i}{\partial q_j}\right) = m\ddot{x}_i \frac{\partial x_i}{\partial q_j} + m\dot{x}_i \underbrace{\frac{d}{dt}\left(\frac{\partial x_i}{\partial q_j}\right)}_{\frac{\partial \dot{x}_i}{\partial q_j}}$

$$= m\ddot{x}_i \frac{\partial x_i}{\partial q_j} + m\dot{x}_i \frac{\partial \dot{x}_i}{\partial q_j}$$

$$\rightarrow m\ddot{x}_i \frac{\partial x_i}{\partial q_j} = \frac{d}{dt}\left(m\dot{x}_i \underbrace{\frac{\partial x_i}{\partial q_j}}\right) - m\dot{x}_i \frac{\partial \dot{x}_i}{\partial q_j}$$

From ⊖ $\frac{\partial x_i}{\partial q_j} = \frac{\partial \dot{x}_i}{\partial \dot{q}_j}$

$$= \frac{d}{dt}\left(m\dot{x}_i \frac{\partial \dot{x}_i}{\partial \dot{q}_j}\right) - m\dot{x}_i \frac{\partial \dot{x}_i}{\partial q_j}$$; Now sum over i.

$$\frac{\partial}{\partial \star}\left(\frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2)\right) = m\dot{x}_1 \frac{\partial \dot{x}_1}{\partial \star} + m\dot{x}_2 \frac{\partial \dot{x}_2}{\partial \star} + m\dot{x}_3 \frac{\partial \dot{x}_3}{\partial \star}$$

$\Rightarrow$

$$\underbrace{\sum_i m\ddot{x}_i \frac{\partial x_i}{\partial q_j}} = \frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_j}\Big(\underbrace{\tfrac{1}{2}mv^2}_{T}\Big)\right) - \frac{\partial}{\partial q_j}\Big(\underbrace{\tfrac{1}{2}mv^2}_{T}\Big)$$

$$= \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j}$$

$$\left[\sum Q_j \delta q_j - \sum_j (\quad)\delta q_j\right] = \emptyset$$

$$\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} - \sum Q_j\right]\delta q_j = \emptyset$$

Assumption: $\vec{F}_a$ can be written as $-\vec{\nabla} u$

$$F_{ai} = -\frac{\partial u}{\partial x_i}$$

$$Q_j = \sum_i F_{ai}\frac{\partial x_i}{\partial q_j} = \sum_i -\frac{\partial u}{\partial x_i}\frac{\partial x_i}{\partial q_j} = -\frac{\partial u}{\partial q_j}$$ {That's it's def$^{\underline{n}}$}.

$$\sum_j \underbrace{\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} - \left(-\frac{\partial u}{\partial q_j}\right)\right]}_{\text{each coef} = \emptyset}\delta q_j = \emptyset$$

independent.

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial}{\partial q_j}(T-u) = \emptyset \quad \text{for each } j.$$

Last assumption: u doesn't depend on $\dot{q}_j$ explicitly.

$$\boxed{\frac{d}{dt}\left(\frac{\partial (T-u)}{\partial \dot{q}_j}\right) - \frac{\partial}{\partial q_j}(T-u) = \emptyset}$$

e.g.: polar coords. no potential

$T = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$ $\quad \vec{v} = \dot{r}\hat{r} + (r\dot{\theta})\hat{\theta}$

$U = \emptyset$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{r}}\right) - \frac{\partial T}{\partial r} = Q_r$$ ← Other forces outside of U.

$\frac{d}{dt}(m\dot{r}) \qquad m r\dot{\theta}^2$

$$\underline{m\ddot{r} - mr\dot{\theta}^2 = Q_r = F_r}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} = Q_\theta$$

$\frac{d}{dt}(mr^2\dot{\theta}) \qquad \emptyset = Q_\theta$

$$2mr\dot{r}\dot{\theta} + mr^2\ddot{\theta} = \tau_{z,\text{axis}} = F_\theta r$$

$$\rightarrow \underline{F_\theta = mr\ddot{\theta} + 2m\dot{r}\dot{\theta}}$$

Same as from Lect. 4.