Theorem 1 (Heine-Borel). *For a given closed interval $I = [a, b]$ in $\mathbb{R}$, if for each point $c \in [a, b]$ there corresponds a number δ_c and an interval I_c of length $2\delta_c$ with c the center point, then there exists a finite number of these intervals which will cover the whole interval $a \leq x \leq b$.*[1] *That is, every point of $a \leq x \leq b$ will be in at least one of the above mentioned finite number of intervals.*[2]

Proof. We begin by first addressing a helpful property of closed intervals which we will present here without proof.

Lemma 1. *If*

$$[a_1, b_1] \supseteq [a_2, b_2] \supseteq [a_3, b_3] \supseteq \cdots$$

is a sequence of nested closed intervals, then

$$\bigcap_{n=1}^{\infty} [a_n, b_n] \neq \varnothing$$

If also $\lim_{n\to\infty}(b_n - a_n) = 0$, *then the infinite intersection consists of a unique real number.*

Now consider our original theorem and suppose that this theorem is false. We now consider the original interval $[a, b]$ and subdivide it into an arbitrary number of subintervals. If, in any of these subintervals, we can find a point $x = c$ such that the corresponding I_c covers the subinterval, then we will consider that subinterval *suitable* (in that it satisfies the conditions of the theorem). If all the subintervals are suitable, then the original interval can be covered by a finite number of subintervals, and our theorem is satisfied.

However, if an interval is not suitable, then we divide it in half and consider the resulting shorter subintervals. If this subdivision process terminates, then we have found a finite number of subintervals. If the process does not terminate, then by Lemma 1, we know for any sequence of subintervals, there exists an $x \in [a, b]$ such that x is the unique real number in the infinite intersection of unsuitable subintervals. However, this violates our original hypothesis and thus, our theorem is proved. □

[1] The converse of this statement is also true.

[2] It is important to note that though not emphasized, this is true for all such δ_c. In other words, for any closed interval I, if we have an infinite cover of I, then we will have a finite cover of I.